**Exact transport equation for local eddy viscosity in turbulent shear flow**

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Two-equation models that treat the transport equations for two variables are typical models for the Reynolds-averaged Navier-Stokes equation. Compared to the equation for the turbulent kinetic energy, the equation for the second variable such as the energy dissipation rate has not been validated enough from the theoretical point of view. In this work, the transport equation for the eddy viscosity was derived and examined for better understanding turbulence and improving turbulence models. The local approximation was first applied to the exact nonlocal eddy-viscosity representation for the Reynolds stress. A new length scale was then introduced, which involves the response function for the velocity fluctuation. It was shown that the local eddy viscosity can be expressed as the correlation between the velocity fluctuation and the new length scale. The exact transport equations for the local eddy viscosity and the length-scale variance were derived theoretically. A direct numerical simulation of turbulent channel flow was carried out to evaluate statistics such as terms in the transport equations. In the local eddy-viscosity equation, the production term due to the Reynolds stress shows large positive values, whereas the pressure–strain correlation term plays a role of destructing the eddy viscosity. The transport equation for the length-scale variance was also examined. It is expected that the analysis of the transport equations is helpful in developing better turbulence models. © 2013 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4816702]

I. INTRODUCTION

The Reynolds-averaged Navier–Stokes (RANS) models have been widely used in practical simulations of turbulent flows. In order to solve the equation for the mean velocity, it is necessary to model the Reynolds stress. The eddy-viscosity representation is often used because of its simple expression. The eddy viscosity can be approximately estimated as \( k^{1/2}L \) where \( k \) is the turbulent kinetic energy and \( L \) is the length scale characterizing the size of the large energy-containing eddies. In two-equation turbulence models, a second equation that determines \( L \) is solved in addition to the \( k \) equation. The second variable does not have to be the length scale \( L \) itself; any combination of the form \( Z = k^mL^n \) is possible because \( k \) is known from its transport equation. The most popular two-equation model is the \( k–\varepsilon \) model\(^1\) that treats the energy dissipation rate \( \varepsilon \) (\( \propto k^{3/2}L^{-1} \)). In the \( k–\omega \) model,\(^2\) the inverse of the turbulent time scale, \( \omega \) (\( \propto k^{1/2}L^{-1} \)) is adopted as the second variable. On the other hand, the quantity \( kL \) related to the two-point velocity correlation is used in the \( k–kL \) model.\(^3,4\)

The transport equation for the turbulent energy is well defined and has been investigated in detail. Terms involved in the transport equation were accurately evaluated using direct numerical simulation (DNS) of various turbulent flows; the data can be used to assess the model expressions including the dissipation and diffusion terms. In contrast, the model equations for the second variables are not well established. The exact transport equation for the dissipation rate \( \varepsilon \) can be derived and evaluated

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using DNS of turbulent channel flow. However, it was shown that the terms in the exact equation do not necessarily correspond to the terms in the model equation because their dependence on the Reynolds number is different. Several attempts have been made to theoretically derive the model equation for \( \varepsilon \). A statistical theory called the two-scale direct interaction approximation was used to derive the \( \varepsilon \) equation. Integration of spectral evolution equations was also used to obtain the \( \varepsilon \) equation. Nevertheless, the model equation for the dissipation rate is not justified theoretically. If a well-defined transport equation for another second variable is derived, it can be used to assess the model equations for \( \varepsilon \), \( \omega \), and \( kL \) using the relationship \( Z = k^m L^n \). The eddy viscosity and diffusivity are candidates for the second variable.

While two-equation models are necessary to predict various types of turbulent flows, one-equation models are used for some turbulent flows for limited area. For aeronautical flows the Spalart-Allmaras model is widely used in which the transport equation for the eddy viscosity is solved. In contrast, Yoshizawa et al. proposed the three-equation model that treats the transport equations for \( k \), \( \varepsilon \), and the eddy viscosity. The third equation was introduced to explicitly deal with the nonstationary and advection effects of the eddy viscosity. Therefore, it must be interesting and important to investigate the transport equation for the eddy viscosity.

Since the eddy diffusivity representation for the scalar flux is simpler than the eddy-viscosity representation for the Reynolds stress, Hamba investigated the transport equation for the eddy diffusivity. A new turbulent length scale associated with the scalar diffusion was introduced and the exact transport equation for the eddy diffusivity was derived. However, the eddy viscosity is more appropriate for examining the transport process of turbulent flow itself. Therefore, in the present work we focus on the eddy viscosity.

Hamba investigated an exact expression for the Reynolds stress using the nonlocal eddy-viscosity representation. The nonlocal eddy viscosity was expressed in terms of the response function for the velocity equation and was evaluated using DNS of channel flow. The analysis of the nonlocal eddy viscosity is complex because it is a two-point two-time correlation; it is very difficult to derive the transport equation for the nonlocal eddy viscosity. Nevertheless, the analysis gives an important clue to the investigation of the eddy viscosity.

In the present work, we focus on the local eddy viscosity obtained from the integral of the nonlocal eddy viscosity. We introduce a new turbulent length scale associated with the momentum transport; the length scale is governed by a transport equation similar to the velocity equation. Since the local eddy viscosity is expressed in terms of the velocity fluctuation and the length scale, its transport equation can be derived in a simple manner. This paper is organized as follows. In Sec. II, we define a turbulent length scale and derive the transport equations for the local eddy viscosity and the length-scale variance. In Sec. III, we evaluate the local eddy viscosity, the length scale, and their transport equations using DNS of channel flow. We investigate the mechanism of the generation of the eddy viscosity and compared the derived equation with existing model equations. Concluding remarks are given in Sec. IV.

II. FORMULATION

In order to solve the transport equation for the mean velocity, it is necessary to model the Reynolds stress. In the eddy-viscosity model, the Reynolds stress is given by

\[
\langle u'_i u'_j \rangle^* = -2 \nu_T S_{ij}, \quad S_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right),
\]

where asterisk indicates the deviatoric part of a tensor \( (a^*_{ij} = a_{ij} - \delta_{ij} a_{kk}/3) \), \( \langle \rangle \) denotes ensemble averaging, \( U_i(\equiv \langle u_i \rangle) \) is the mean velocity, \( u'_i \) is the velocity fluctuation, and \( \nu_T \) is the eddy viscosity. This approximation is widely used for practical turbulence models.

In the nonlinear eddy-viscosity model, the Reynolds stress is expressed as

\[
\langle u'_i u'_j \rangle^* = -2 \nu_T S_{ij} + \xi_1 (S_{ik} S_{kj})^* + \xi_2 (S_{ik} \Omega_{kj} - \Omega_{ik} S_{kj}) + \xi_3 (\Omega_{ik} \Omega_{kj})^*,
\]

where \( \Omega \) is the vorticity tensor.
where

$$\Omega_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i} \right),$$

(3)

and the summation convention is used for repeated indices. The second to fourth terms involving $\zeta_i$ ($i = 1-3$) on the right-hand side of Eq. (2) are the nonlinear eddy-viscosity terms introduced to improve the eddy-viscosity model.\(^\text{17,18}\) In addition to the quadratic nonlinear terms appearing in Eq. (2), cubic nonlinear terms are necessary to predict a wide range of three-dimensional flows.\(^\text{19}\)

Equation (2) can be rewritten as

$$\langle u'_i u'_j \rangle^* = -\nu \frac{\partial U_k}{\partial x_m},$$

(4)

where $\nu_{ijkm}$ is the fourth-rank eddy-viscosity tensor that can explicitly depend on the mean velocity gradient.

The eddy-viscosity model is local in space in the sense that the Reynolds stress at a point is expressed in terms of physical quantities at the same point. This local approximation is valid only if the turbulence length scale is much less than the length scale of the mean field variation.\(^\text{20}\) However, this condition does not always hold for actual turbulent flows. A few nonlocal models for the Reynolds stress were investigated. For example, Nakayama\(^\text{21}\) proposed the following nonlocal model for the Reynolds stress:

$$\langle u'_i u'_j \rangle(y) = -\int dy' \nu_{ijL}(y; y') \frac{\partial U_i(y')}{\partial y'},$$

(5)

where $\nu_{ijL}$ is a coefficient representing a nonlocal contribution, which is hereafter called the nonlocal eddy viscosity. The above model assumes that the Reynolds stress at the $y$ location is affected by the $y'$ location.

Using the response function for the velocity fluctuation, Hamba\(^\text{16}\) theoretically derived an exact nonlocal expression for the Reynolds stress as follows:

$$\langle u'_i u'_j \rangle(x, t) = -\int dx' \int_0^t dt' \nu_{ijLijkm}(x, t; x', t') \frac{\partial U_k(x', t')}{\partial x_m},$$

(6)

where

$$\nu_{ijLijkm}(x, t; x', t') = \langle u'_i(x, t) g_{jk}(x, t; x', t') u'_m(x', t') \rangle.$$  

(7)

The nonlocal eddy viscosity $\nu_{ijLijkm}(x, t; x', t')$ is a coefficient representing a nonlocal contribution in space and in time; it is expressed as the correlation between the velocity fluctuations and the response function $g_{jk}$. The expression can be derived as follows. First, we consider the Navier-Stokes and continuity equations for the velocity fluctuation given by

$$\frac{Du'_j}{Dt} + \frac{\partial}{\partial x_k} (u'_i u'_{j}) + \frac{\partial p'}{\partial x_j} - \nu \frac{\partial^2 u'_j}{\partial x_k \partial x_k} = -u'_k \frac{\partial U_j}{\partial x_k},$$

(8)

where $D/Dt = \partial/\partial t + U_i \partial/\partial x_i$, $p'$ is the pressure fluctuation, and $\nu$ is the molecular viscosity. We formally consider the right-hand side of Eq. (8) an external force term for the velocity fluctuation. We then introduce the response function $g_{jk}(x, t; x', t')$ satisfying the following equation:

$$\frac{Dg_{jk}}{Dt} + \frac{\partial}{\partial x_m} (u'_m g_{jk} - \langle u'_m g_{jk} \rangle) + \frac{\partial p_{jk}}{\partial x_j} - \nu \frac{\partial^2 g_{jk}}{\partial x_m \partial x_m} = \delta_{jk} \delta(x - x') \delta(t - t'),$$

(10)

where $\delta$ is the Dirac delta function.
Using this response function, the formal solution for the velocity fluctuation can be written as

$$u'_j(x, t) = -\int d\mathbf{x}' \int_0^t dt' g_{jk}(x, t; \mathbf{x}', t') u'_m(\mathbf{x}', t') \frac{\partial}{\partial x'_m} U_k(\mathbf{x}', t').$$ \hspace{1cm} (12)

Here, terms involving the initial value $u'_j(x, 0)$ are omitted because they do not contribute to the Reynolds stress at time $t$ when $t$ is sufficiently large. Multiplying Eq. (12) by $u'_j$ and taking ensemble averaging, we obtain Eqs. (6) and (7) for the Reynolds stress and the nonlocal eddy viscosity.

The expression for the nonlocal eddy viscosity was validated using DNS of turbulent channel flow. However, the evaluation of the nonlocal eddy viscosity is time consuming because it is a two-point two-time correlation. Moreover, it seems difficult to model the nonlocal eddy viscosity in a straightforward manner. We then focus attention on the local eddy viscosity; it can be derived from the nonlocal eddy viscosity as follows. In general, the nonlocal eddy viscosity $\nu_{NLijkm}$ has a nonzero value if the distance $|\mathbf{x} - \mathbf{x}'|$ and the time difference $t - t'$ are comparable with or less than the turbulence length and time scales, respectively. If in this region in space and time the velocity gradient $\partial U_k/\partial x'_m$ is nearly uniform, the Reynolds stress can be approximated by

$$\langle u'_j u'_j \rangle = -\nu_{Lijkm} \frac{\partial U_k}{\partial x'_m},$$ \hspace{1cm} (13)

where $\nu_{Lijkm}$ is the local eddy viscosity defined as

$$\nu_{Lijkm}(x, t) \equiv \int d\mathbf{x}' \int_0^t dt' \nu_{NLijkm}(x, t; \mathbf{x}', t')$$

$$= \int d\mathbf{x}' \int_0^t dt' \langle u'_j(x, t) g_{jk}(x, t; \mathbf{x}', t') u'_m(\mathbf{x}', t') \rangle.$$ \hspace{1cm} (14)

A similar approximation to the integration is used to derive the rapid term of the pressure–strain correlation appearing in the Reynolds-stress equation model, in which the mean velocity gradient is assumed uniform in the integral region. Whether the local approximation (13) is good or not depends on the relationship in the length and time scales between profiles of $\nu_{NLijkm}$ and $\partial U_k/\partial x'_m$.

Although Eq. (13) is a local expression, the local eddy viscosity given by Eq. (14) still contains two-point two-time quantities; it is necessary to evaluate the nonlocal eddy viscosity and integrate it. In the present work, we introduce the following quantity that has a dimension of length scale:

$$\ell'_{jkm}(x, t) = \int d\mathbf{x}' \int_0^t dt' g_{jk}(x, t; \mathbf{x}', t') u'_m(\mathbf{x}', t').$$ \hspace{1cm} (15)

Using the new length scale $\ell'_{jkm}$, the local eddy viscosity can be simply written as

$$\nu_{Lijkm} = \langle u'_j \ell'_{jkm} \rangle.$$ \hspace{1cm} (16)

Furthermore, multiplying each term in Eq. (10) by $u'_m(\mathbf{x}', t')$ and integrating it with respect to $\mathbf{x}'$ and $t'$, we can obtain the transport equations for $\ell'_{jkm}$ as follows:

$$\frac{D \ell'_{jkm}}{Dt} + \frac{\partial}{\partial x_n} \left( u'_n \ell'_{jkm} - \langle u'_n \ell'_{jkm} \rangle \right) + \frac{\partial p'_{Lkm}}{\partial x_j} - \nu \frac{\partial^2 \ell'_{jkm}}{\partial x_n \partial x_n} = u'_m \delta_{jk},$$ \hspace{1cm} (17)

$$\frac{\partial \ell'_{jkm}}{\partial x_j} = 0.$$ \hspace{1cm} (18)

We should note that Eqs. (17) and (18) do not contain quantities depending on $\mathbf{x}'$ and $t'$. Therefore, we can evaluate the length scale $\ell'_{jkm}$ by solving the transport equation without using the response function or the velocity at different position and time; we can evaluate the local eddy viscosity given by Eq. (16) in a simple manner.

Here, let us discuss physical meaning of the length scale $\ell'_{jkm}$. As mentioned in Sec. I, the eddy viscosity is expressed qualitatively as the product of the turbulent intensity $k^{1/2}$ and the turbulent length scale $L$. In Eq. (16), the eddy viscosity is defined qualitatively as the correlation between the
velocity fluctuation $u'_i$ and the length scale $\ell'_{jkm}$. The length scale $\ell'_{jkm}$ can be considered a kind of mixing length for the momentum transport.\textsuperscript{22} If the local approximation is valid, the formal solution (12) of the velocity fluctuation can be written as

$$u'_j = -\ell'_{jkm} \frac{\partial U_k}{\partial x_m}.$$  \hspace{1cm} (19)

Figure 1 shows a schematic plot of the mixing length. When a fluid parcel moves from $y$ to $y + \ell'_y$ keeping its characteristics, the value of the velocity $u_x$ becomes less than that of the surrounding fluid. The velocity increment $u'_x$ is equal to minus the product of the mixing length $\ell'_y$ and the mean velocity gradient; this relationship agrees with Eq. (19). This is a qualitative explanation of the length scale $\ell'_{jkm}$; its value can be obtained by solving the transport equation (17).

Using Eqs. (17) and (8), we can derive the transport equation for the local eddy viscosity as follows:

$$\frac{D}{Dt}(u'_i \ell'_{jkm}) = (u'_i u'_m) \delta_{jk} - (u'_n \ell'_{jkm}) \frac{\partial U_l}{\partial x_n} - 2\nu \left( \frac{\partial u'_i}{\partial x_n} \frac{\partial \ell'_{jkm}}{\partial x_m} + \frac{p'}{\ell'_{jkm}} \frac{\partial \ell'_{jkm}}{\partial x_j} \right)$$

$$- \frac{\partial}{\partial x_n} \left( (u'_n u'_j \ell'_{jkm}) + (p' \ell'_{jkm}) \delta_{jn} + (p'_{Lkm} u'_i) \delta_{jn} - \nu \frac{\partial}{\partial x_n} (u'_i \ell'_{jkm}) \right).$$  \hspace{1cm} (20)

The right-hand side consists of five terms. The first and second terms are production terms; the first term shows that the turbulent velocity fluctuations generate the eddy viscosity, whereas the second term represents the anisotropy effect due to the mean velocity gradient. The third term is the dissipation term due to the molecular viscosity. The fourth term is the correlation between the pressure and the length/velocity gradient; hereafter we call it the pressure–strain term. This term is expected to play a role of destructing the eddy viscosity. The fifth term is the diffusion term involving the turbulent and viscous fluxes. We should note that the first production term is the Reynolds stress itself and it does not accompany the mean velocity gradient; even if there is no mean shear, the turbulent fluctuations can generate the eddy viscosity.

Since the length scale $\ell'_{jkm}$ vanishes when averaged, the variance $\langle \ell'^2_{jkm} \rangle$ is an important statistics (here the summation convention is not applied to $i, j, k, m$). Using Eq. (17), we can derive the transport equation for $\langle \ell'^2_{jkm} \rangle$ as follows:

$$\frac{D}{Dt}(\ell'^2_{jkm}) = 2(u'_n \ell'_{jkm}) \delta_{jk} - 2\nu \left( \frac{\partial \ell'_{jkm}}{\partial x_n} \frac{\partial \ell'_{jkm}}{\partial x_m} + \frac{p'_{Lkm} \ell'_{jkm}}{\partial x_j} \right)$$

$$- \frac{\partial}{\partial x_n} \left( (u'_n \ell'^2_{jkm}) + 2(p'_{Lkm} \ell'_{jkm}) \delta_{jn} - \nu \frac{\partial}{\partial x_n} (\ell'^2_{jkm}) \right).$$  \hspace{1cm} (21)
The right-hand side consists of four terms: the production, dissipation, pressure–strain, and diffusion terms. In this case, the production term is the local eddy viscosity itself. Therefore, we can see that the length scale, the eddy viscosity, and the Reynolds stress are closely related to each other through the production terms.

III. ESTIMATE USING CHANNEL FLOW DNS

In Sec. II, introducing the new length scale \( \ell_{jkm} \), we obtained the expression for the local eddy viscosity. We also derived the exact transport equations for the local eddy viscosity \( [u'_i \ell_{jkm}] \) and the length-scale variance \( \langle \ell_{jkm}^2 \rangle \). In this section, using DNS of turbulent channel flow, we investigate the local eddy viscosity and the transport equations.

A. Numerical method

In the DNS, we numerically solve the equations for the velocity given by

\[
\frac{\partial u_i}{\partial t} = -\frac{\partial}{\partial x_j} (u_j u_i) - \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + f_i \delta_{ij},
\]

(22)

\[
\frac{\partial u_i}{\partial x_i} = 0,
\]

(23)

where \( f_i \) is an external force. The variables \( x_1(x) \), \( x_2(y) \), and \( x_3(z) \) denote the coordinates in the streamwise, wall-normal, and spanwise directions, respectively; the corresponding velocity components are given by \( u_1(=u_x) \), \( u_2(=u_y) \), and \( u_3(=u_z) \). For the channel flow the non-zero component of the mean velocity gradient is \( \partial U_j/\partial y \). Therefore, the local eddy viscosity \( v_{Lijxy} = [u'_i \ell_{jxy}] \) is necessary in Eq. (13) and the length scale \( \ell_{jxy} \) needs to be calculated. For simplicity, \( \ell_{jxy} \) and \( \ell_{Lxy} \) are denoted by \( \ell'_j \) and \( \ell'_L \), respectively. In addition to the velocity, the transport equations for the length scale \( \ell'_j \) are solved as follows:

\[
\frac{\partial \ell'_j}{\partial t} = -\frac{\partial}{\partial x_k} (u_k \ell'_j) - \frac{\partial p'_L}{\partial x_j} + \nu \frac{\partial^2 \ell'_j}{\partial x_k \partial x_k} + u'_i \delta_{ij},
\]

(24)

\[
\frac{\partial \ell'_L}{\partial x_j} = 0.
\]

(25)

Hereafter, all quantities are nondimensionalized by the wall-friction velocity \( u_\tau = (\nu \partial U_j/\partial y)_{wall} \) and the channel half width \( L_c/2 \).

The Reynolds number based on the friction velocity is set to \( Re_\tau = u_\tau (L_c/2)/\nu = 590 \). The size of the computational domain is \( L_c \times L_c \times L_c = 2\pi \times 2\pi \times \pi \). A staggered grid is adopted; it is uniform in the \( x \) and \( z \) directions and is stretched in the \( y \) direction using a hyperbolic function. The number of the grid points is \( N_c \times N_c \times N_c = 1024 \times 192 \times 1024 \). The periodic boundary conditions for \( u_i \) and \( \ell'_j \) are used in the \( x \) and \( z \) directions. No-slip conditions \( u_i = 0 \) and \( \ell'_j = 0 \) are imposed at the walls (\( y = \pm 1 \)). We use the second-order finite-difference scheme in space and the Adams-Bashforth method for time marching. The computational time step is \( \Delta t = 10^{-4} \). Statistics such as the eddy viscosity were accumulated over a time period of 30 normalized by \( (L_c/2)u_\tau \).

Here, we should mention the averaged term involved on the right-hand side of Eq. (24). Since we take an average over the \( x-z \) plane and in time to obtain statistics, the term \( \langle u_k \ell'_j \rangle \) cannot be evaluated until the time marching is finished. Instead, in this work, the average on the right-hand side is approximated by that over the \( x-z \) plane only.

Since the velocity and length-scale fields are homogeneous in the \( x \) and \( z \) directions, statistics depend only on \( y \). Figure 2 shows the mean velocity profile as a function of \( y \). The mean velocity obtained from DNS by Moser et al.\(^{23} \) is also shown; the present result agrees well with their data. It is clearly seen that the velocity gradient is steep near the wall at \(-1 < y < -0.9 \) and at \( 0.9 < y < 1 \), whereas the profile is fairly flat apart from the wall at \(-0.9 < y < 0.9 \).
Figure 2. Profile of mean velocity $U_x$. The profile obtained from the DNS by Moser et al.\textsuperscript{23} is also plotted.

Figure 3 shows the profiles of the rms velocity fluctuations $\sqrt{\langle u'^2_x \rangle}$, $\sqrt{\langle u'^2_y \rangle}$, and $\sqrt{\langle u'^2_z \rangle}$. The results of DNS by Moser et al.\textsuperscript{23} are also shown for comparison. Although the profiles of the velocity fluctuations of channel flow are known very well, they are plotted here because they play an important role in the transport equation for the eddy viscosity. In the whole region, the streamwise component is greater than the other components. This anisotropy appears because the streamwise component of the turbulent energy is first generated because of the mean shear and the energy is transferred to the other components via the pressure–strain term. In the near-wall region, the anisotropy is very strong because the energy redistribution due to the pressure fluctuation decreases owing to the wall-blocking effect.

B. Local approximation

We assumed the local approximation in deriving Eq. (13). Here, we examine the validity of the local approximation for the Reynolds stress. In order to solve the equation for the mean velocity $U_x$, the shear stress $\langle u'_x u'_y \rangle$ needs to be modeled. Equation (13) for the shear stress can be written in the following two forms:

$$\langle u'_x u'_y \rangle_A = -\langle u'_x \ell'_y \rangle \frac{\partial U_x}{\partial y},$$  

(26)

$$\langle u'_x u'_y \rangle_B = -\langle u'_y \ell'_x \rangle \frac{\partial U_x}{\partial y}. \quad (27)$$

In Eq. (26), the wall-normal velocity component is replaced by the length scale and the mean velocity gradient, whereas the streamwise velocity component is replaced similarly in Eq. (27). Figure 4(a)

FIG. 2. Profile of mean velocity $U_x$. The profile obtained from the DNS by Moser et al.\textsuperscript{23} is also plotted.

FIG. 3. Profiles of the rms velocity fluctuations $\sqrt{\langle u'^2_x \rangle}$, $\sqrt{\langle u'^2_y \rangle}$, and $\sqrt{\langle u'^2_z \rangle}$. The results of DNS by Moser et al.\textsuperscript{23} are also plotted.
FIG. 4. Profiles of turbulent shear stresses $\langle u'_x u'_y \rangle_A$, $\langle u'_x u'_y \rangle_B$, and $\langle u'_x u'_y \rangle$ for (a) $Re_\tau = 590$ and (b) $Re_\tau = 395$.

shows the profiles of the shear stresses $\langle u'_x u'_y \rangle_A$ and $\langle u'_x u'_y \rangle_B$ in addition to the profile of $\langle u'_x u'_y \rangle$ directly evaluated from the DNS. The magnitude of $\langle u'_x u'_y \rangle_A$ is too large near the wall, whereas the profile of $\langle u'_x u'_y \rangle_B$ agrees fairly well with the directly obtained value. This difference can be explained as follows. The formal solution for $u'_i$ given by Eq. (12) represents the influence of the velocity gradient $\partial U_k/\partial x_m$ on $u'_i$. The influence of $\partial U_x/\partial y$ on the streamwise component $u'_x$ is direct in the sense that the transport equation for $u'_x$ involves the mean velocity term $-u'_y \partial U_x/\partial y$; the mean shear directly generates the streamwise velocity fluctuation at the same position. Therefore, the local approximation of $\langle u'_x u'_y \rangle_B$ is good. In contrast, the influence of $\partial U_x/\partial y$ on the wall-normal component $u'_y$ is not direct because the transport equation for $u'_y$ does not involve the mean velocity term. The streamwise fluctuation generated by the mean shear is redistributed to the wall-normal fluctuation via the pressure-gradient term. The pressure fluctuation obeys the Poisson equation and shows nonlocal behavior in space. Therefore, the influence of $\partial U_x/\partial y$ on $u'_y$ is nonlocal and the local approximation of $\langle u'_x u'_y \rangle_A$ is not good.

At present, it is not clear how the above discussion is applicable to more complicated shear flow cases. A linear combination of $\langle u'_x u'_y \rangle_A$ and $\langle u'_x u'_y \rangle_B$ is a candidate for an estimate in general cases. The production term $-\langle u'_i u'_k \rangle \partial U_j/\partial x_k - \langle u'_i u'_k \rangle \partial U_j/\partial x_k$ in the $\langle u'_x u'_y \rangle$ equation may be helpful for the estimate. The above situation for the simple shear case corresponds to the fact that the first part of the production vanishes, whereas the second part has a positive value. Therefore, we expect that the value of the production term can be used to derive appropriate coefficients of the linear combination.

In this paper, we discuss the comparison with DNS for $Re_\tau = 590$. To show the dependence of the results on the Reynolds number, we plot the shear stresses for $Re_\tau = 395$ in Fig. 4(b). Since the Reynolds number is small, the location of the peak of the shear stress is slightly far from the wall compared to the case of $Re_\tau = 590$. The profiles of the two local approximations for $Re_\tau = 395$ are similar to the corresponding ones for $Re_\tau = 590$; their dependence on the Reynolds number is small. It remains as future work to compare with higher-Reynolds-number flows.

FIG. 5. Profiles of the eddy viscosities $\langle u'_x u'_y \rangle$, $v_E$, $0.09k^2/\epsilon$, and $0.2\langle u'_x^2 \rangle k/\epsilon$. 
We then examine the local eddy viscosity \( v_{L_{xx}}(=\langle u'_x \ell'_x \rangle) \) for the shear stress \( \langle u'_x u'_x \rangle_B \). For comparison, the effective eddy viscosity is defined as

\[
v_E = -\langle u'_x u'_x \rangle \int \frac{\partial U_x}{\partial y}.
\]  

(28)

If the local approximation for \( \langle u'_x u'_x \rangle_B \) is valid, the local eddy viscosity \( \langle u'_x \ell'_x \rangle \) should agree with the effective eddy viscosity \( v_E \). Figure 5 shows the profiles of the eddy viscosities \( \langle u'_x \ell'_x \rangle \) and \( v_E \). The value of \( v_E \) shows a fluctuation at the channel center because the numerator and denominator in Eq. (28) are very small. The two eddy viscosities nearly agree with each other near the wall, whereas \( \langle u'_x \ell'_x \rangle \) is slightly less than \( v_E \) apart from the wall. The eddy viscosity for the standard \( k-\varepsilon \) model is given by

\[
v_T = C_v \frac{k^2}{\varepsilon},
\]  

(29)

where \( k(=\langle u'^2 \rangle/2) \) is the turbulent kinetic energy, \( \varepsilon \) is its dissipation rate, and \( C_v(=0.09) \) is a nondimensional model constant. The profile of this eddy viscosity is also plotted in Fig. 5 for comparison. The local eddy viscosity \( \langle u'_x \ell'_x \rangle \) is closer to the effective eddy viscosity \( v_E \) than the eddy viscosity for the \( k-\varepsilon \) model is. Therefore, the local approximation of \( \langle u'_x u'_x \rangle_B \) with the local eddy viscosity \( \langle u'_x \ell'_x \rangle \) is good for the channel flow.

The good profile of \( \langle u'_x u'_x \rangle_B \) suggests that the use of \( u'_x \) component is appropriate to the wall-normal mixing. This situation is closely related to the \( v^2-f \) formula proposed by Durbin.24 In the \( v^2-f \) model, the eddy viscosity is expressed as

\[
v_T = C'_v \langle u'^2 \rangle \frac{k^2}{\varepsilon},
\]  

(30)

where \( C'_v(=0.2) \) is a nondimensional model constant. The profile of this eddy viscosity is also plotted in Fig. 5 for comparison. Its profile agrees well with \( \langle u'_x \ell'_x \rangle \) and \( v_E \) in the near-wall region.

As shown in Fig. 3, the anisotropy of the normal stresses is an important property of shear flows. The nonlinear eddy-viscosity model given by Eq. (2) can predict the anisotropy. The deviatoric parts of the normal stresses, \( \langle u'^2 \rangle^* = \langle u'^2 \rangle - (2/3)k \), are written as

\[
\langle u'^2 \rangle_x^* = \left[ \frac{1}{12} (\zeta_1 - \zeta_3) - \frac{1}{2} \xi_2 \right] \left( \frac{\partial U_x}{\partial y} \right)^2,
\]  

(31)

\[
\langle u'^2 \rangle_y^* = \left[ \frac{1}{12} (\zeta_1 - \zeta_3) + \frac{1}{2} \xi_2 \right] \left( \frac{\partial U_y}{\partial y} \right)^2,
\]  

(32)

\[
\langle u'^2 \rangle_z^* = -\frac{1}{6} (\zeta_1 - \zeta_3) \left( \frac{\partial U_z}{\partial y} \right)^2.
\]  

(33)

The difference between the normal components for simple shear flow cannot be reproduced by the isotropic eddy viscosity \( v_T \), but by the nonlinear terms involving \( \zeta_i \). The normal components can also be expressed in terms of the local eddy-viscosity representation given by Eq. (13) as follows:

\[
\langle u'^2 \rangle_x^*_A = -v_{L_{xx}}^* \frac{\partial U_x}{\partial y},
\]  

(34)

\[
\langle u'^2 \rangle_y^*_A = -v_{L_{yy}}^* \frac{\partial U_y}{\partial y},
\]  

(35)

\[
\langle u'^2 \rangle_z^*_A = -v_{L_{zz}}^* \frac{\partial U_z}{\partial y}.
\]  

(36)

In this case, the eddy-viscosity tensor \( v_{L_{xx}} = \langle u'_x \ell'_x \rangle \) explicitly depends on the mean velocity gradient (the summation convention is not applied to \( \alpha \)). Figure 6 shows the profiles of the deviatoric parts of the normal stresses \( \langle u'^2 \rangle^*_A, \langle u'^2 \rangle^*_A, \langle u'^2 \rangle^*_A \) in addition to the corresponding stresses \( \langle u'^2 \rangle^* \).
FIG. 6. Profiles of the deviatoric parts of the normal stresses \( \langle u' \rangle^2_A, \langle u' \rangle^2_y, \langle u' \rangle^2_z, \langle u' \rangle^2_x \), and \( \langle u' \rangle^2 \).

\( \langle u'^2 \rangle_y \) directly evaluated from the DNS. For each stress, the local approximation \( \langle u'^2 \rangle_A \) agrees well with \( \langle u'^2 \rangle \) apart from the wall at \( y > -0.9 \). However, the magnitudes of \( \langle u'^2 \rangle_A \) and \( \langle u'^2 \rangle_y \) are overpredicted near the wall at \( y < -0.9 \). These profiles show that the local approximation is not good near the wall. The anisotropy of the normal stresses is closely related to the pressure fluctuation as discussed in Sec. III A. The pressure fluctuation in the near-wall region is affected by the wall; this is why the local approximation for the normal stresses is not good near the wall.

In summary, for the shear stress, the local approximation given by \( \langle u'u' \rangle_B \) is good, whereas the magnitude of \( \langle u'^2 \rangle_A \) is too large near the wall. For the normal stresses, the magnitude of \( \langle u'^2 \rangle_A \) agrees well with \( \langle u'^2 \rangle \) apart from the wall, but it is overpredicted near the wall. Hereafter, we focus attention on the local eddy viscosity \( \langle u' \ell' \rangle \) for the shear stress \( \langle u'u' \rangle_B \).

C. Transport equation for eddy viscosity

Since the local eddy viscosity involves \( \ell'_i \), the behavior of the length scale should also be examined. The mean value \( \langle \ell'_i \rangle \) vanishes as already mentioned; we evaluate the variance \( \langle \ell'^2 \rangle \) here. Figure 7 shows the profiles of the rms length scales \( \sqrt{\langle \ell'^2_x \rangle}, \sqrt{\langle \ell'^2_y \rangle}, \sqrt{\langle \ell'^2_z \rangle} \). The length scales approximately represent the mixing length, or the velocity fluctuation induced by unit mean shear as suggested by Eq. (19). In contrast to the rms velocity fluctuation whose peak is located near the wall, the rms length scales are large near the channel center. The peak of \( \sqrt{\langle \ell'^2_x \rangle} \) is expected to be located at the channel center; it is slightly deviated from the center because the time period for averaging is not long enough for this component. The value of the streamwise component \( \sqrt{\langle \ell'^2_x \rangle} \) is the largest of the three components. The wall-normal component \( \sqrt{\langle \ell'^2_y \rangle} \) gradually increases near the wall and is as large as the spanwise component \( \sqrt{\langle \ell'^2_z \rangle} \) at the channel center.

FIG. 7. Profiles of the rms length scales \( \sqrt{\langle \ell'^2_x \rangle}, \sqrt{\langle \ell'^2_y \rangle}, \sqrt{\langle \ell'^2_z \rangle} \).
In this model the eddy viscosity is given by

\[ \nu_T = \bar{\nu} f_{1,1}, \]  

where the right-hand side consists of four terms: the first production term due to the Reynolds stress, the dissipation term, the pressure–strain term, and the diffusion term. The second production term with the mean velocity gradient appearing in Eq. (20) vanishes for this eddy viscosity. Figure 8 shows the profiles of terms in the transport equation for \( \langle u_x' \ell_x' \rangle \). The first production term due to the Reynolds stress \( \langle u_x^2 \rangle \) shows large positive values except for the region very close to the wall. Among negative terms, the pressure–strain term is dominant and the dissipation term is negligible. This situation is similar to negative terms in the transport equation for the shear stress \( \langle u_x' u_y' \rangle \). We should note that the first production term decreases as \( y \) increases at \( y > -0.9 \), but it still shows a finite value at the channel center. This finite value is in contrast to the production term in the Reynolds stress equation, in which the term vanishes at the channel center because of zero mean shear. The profile of the production term in Fig. 8 can account for the profile of the eddy viscosity in Fig. 5 in which the peak is located apart from the wall unlike the rms velocity fluctuation.

Since the eddy viscosity \( \langle u_x' \ell_x' \rangle \) involves the length scale \( \ell_x' \), it is interesting to examine the transport equation for \( \langle \ell_x^2 \rangle \). The transport equation can be written for channel flow as follows:

\[ \frac{\partial}{\partial t} \langle \ell_x^2 \rangle = 2 \langle u_x' \ell_x' \rangle - 2 \nu \left( \frac{\partial \ell_x'}{\partial x} \frac{\partial u_x'}{\partial y} \right) + 2 \left( p'_L \frac{\partial \ell_x'}{\partial x} \right) - \frac{\partial}{\partial y} \left( \langle u_x' \ell_x' \rangle + \langle p' \ell_x' \rangle - \nu \frac{\partial}{\partial y} \langle u_x' \ell_x' \rangle \right), \]  

where the right-hand side consists of the production, dissipation, pressure–strain, and diffusion terms. Figure 9 shows the profiles of terms in the transport equation for \( \langle \ell_x^2 \rangle \). The production term due to the eddy viscosity \( \langle u_x' \ell_x' \rangle \) is dominant among positive terms. In contrast to the eddy-viscosity equation, not only the pressure–strain term but also the dissipation term contribute as negative terms. The production term shows a fairly large value even at the channel center. This is why the rms length scale shows a maximum near the channel center in contrast to the rms velocity fluctuation. The length-scale variance \( \langle \ell_x^2 \rangle \) is generated by \( \langle u_x' \ell_x' \rangle \) while the eddy viscosity \( \langle u_x' \ell_x' \rangle \) is generated by \( \langle u_x^2 \rangle \). As discussed in Sec. II, it is shown that the length scale, the eddy viscosity, and the Reynolds stress are closely related to each other through the production terms.

Here, comparing with Eq. (20), we discuss existing model equations for the eddy viscosity. Spalart-Allmaras model\(^{13}\) is the one-equation model treating the transport equation for a variable \( \bar{\nu} \). In this model the eddy viscosity is given by
where $f_{v1}$ is a nondimensional function whose value tends to unity for fully developed turbulence; the model variable $\tilde{\nu}$ is effectively the same as the eddy viscosity apart from the wall where $\tilde{\nu} \gg \nu$ and $f_{v1} \simeq 1$. We should note that the following comparison is not necessarily valid in the near-wall region where $f_{v1} < 1$. The transport equation for $\tilde{\nu}$ is written as

$$
\frac{D\tilde{\nu}}{Dt} = c_{b1}\tilde{S}\tilde{v} - c_{w1}f_{w}\left(\frac{\tilde{v}}{d}\right)^2 + \frac{1}{\sigma} \frac{\partial}{\partial x_i} \left[(\nu + \tilde{\nu}) \frac{\partial \tilde{\nu}}{\partial x_i}\right] + \frac{c_{b2}}{\sigma} \frac{\partial \tilde{\nu}}{\partial x_i} \frac{\partial \tilde{\nu}}{\partial x_i},
$$

(40)

where $\tilde{S}$ is effectively the magnitude of the mean vorticity, $d$ is the distance from the nearest wall, $f_{w}$ is a nondimensional function, and $C_{b1}, C_{w1}, C_{b2}$, and $\sigma$ are nondimensional constants. The first term on the right-hand side of Eq. (40) is the production term. This term is proportional to $\tilde{S}$ that is equal to the mean velocity gradient $\partial U_y/\partial y$ apart from the wall for simple shear flow. Our result for the eddy-viscosity equation (37) for shear flow shows that the production term is given by the Reynolds stress $\langle u'^{2}y \rangle$ itself; we can further express the Reynolds stress $\langle u'^{2}y \rangle$ in terms of $(\partial U_y/\partial y)^2$ taking into account the nonlinear eddy-viscosity model given by Eq. (32). Therefore, the present result suggests that $\tilde{S}^2$ is more appropriate than $\tilde{S}$ for the production term. The second term on the right-hand side of Eq. (40) can be considered a model for the pressure–strain term appearing in Eq. (20). The fourth term on the right-hand side of Eq. (40) is an additional term that is originated from the diffusion term of the dissipation rate equation. The present result suggests that the diffusion term for the $\tilde{\nu}$ equation can be expressed as the divergence of the flux like the turbulent energy equation.

In contrast to the one-equation model, Yoshizawa et al.\textsuperscript{14} proposed the three-equation model that treats the transport equations for $k$, $\varepsilon$, and the eddy viscosity. The third equation was introduced to explicitly deal with the nonstationary and advection effects of the eddy viscosity. The transport equation for the eddy viscosity is given by

$$
\frac{D\nu_T}{Dt} = C_{vP}k - C_{v\nu} \frac{\nu_T}{\tau} + \frac{\partial}{\partial x_i} \left[(\nu + \nu_T) \frac{\nu_T}{\sigma_{\nu}} \frac{\partial \nu_T}{\partial x_i}\right],
$$

(41)

where $\tau$ is the turbulent time scale and $C_{vP}$, $C_{v\nu}$, and $\sigma_{\nu}$ are nondimensional constants. The right-hand side of Eq. (41) consists of the production, dissipation, and diffusion terms. Each term has its counterpart in the local eddy-viscosity equation given by Eq. (20) although there is no production term due to the mean shear in Eq. (41). Let us examine the production term in Eq. (41) in detail; it is proportional to the turbulent energy $k$ and its coefficient is set to $C_{vP} = 4/15$. The eddy viscosity $\nu_T$ treated in the model can be considered the isotropic part of $\nu_{Tijkm} = \nu_{Tijkm} - \nu_{Lankm}\delta_{ij}/3$ in the present analysis. If the turbulent velocity field is isotropic, the eddy-viscosity tensor should be written as

$$
\nu_{Tijkm} = \nu_T(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}) - \frac{2}{3} \nu_T \delta_{ij}\delta_{km}.
$$

(42)
From Eq. (42), the coefficient $v_T$ can be obtained as
\[
v_T = \frac{1}{20} (v_{L_{ij}ij} + v_{L_{ji}ij}) - \frac{1}{30} v_{L_{ij}ij}.
\] (43)

In a similar manner, the isotropic part of the first production term $\langle u'_i u'_m \rangle \delta_{ji}$ in Eq. (20) is given by
\[
\frac{1}{20} \left( \langle u'_i u'_j \rangle \delta_{ji} + \langle u'_j u'_i \rangle \delta_{ij} \right) - \frac{1}{30} \langle u'_i u'_i \rangle \delta_{ij} = \frac{1}{6} \langle u'_1^2 \rangle = \frac{1}{3} k,
\] (44)

which is proportional to $k$ and its coefficient is close to $C_{\nu P} = 4/15$. Therefore, our result can account for the production term in Eq. (41). At the same time, it is expected that the second production term involving the mean velocity gradient can be added to Eq. (41) in order to improve the three-equation model.

**IV. CONCLUSIONS**

The exact transport equation for the local eddy viscosity was derived and examined for better understanding turbulence and improving turbulence models. The nonlocal eddy-viscosity representation for the Reynolds stress was first examined and the local approximation was applied. A new length scale was introduced that involves the response function for the velocity fluctuation. This length scale can be considered a mixing length for the momentum transport. It was shown that the local eddy viscosity can be expressed as the correlation between the velocity fluctuation and the new length scale. The governing equation for the length scale was also obtained. Using the governing equation, we derived the exact transport equations for the local eddy viscosity and the length-scale variance.

A DNS of turbulent channel flow was carried out to evaluate statistics including the local eddy viscosity. The validity of the local approximation for the eddy viscosity was examined. Two forms are possible for the local eddy viscosity of the shear stress; it was shown that the local approximation is good for the form in which the effect of the mean shear on the streamwise velocity fluctuation is taken into account. Terms in the transport equation for the local eddy viscosity of the shear stress were evaluated. The production term due to the Reynolds stress shows large positive values, whereas the pressure–strain term plays a role of destructing the eddy viscosity. In the transport equation for the length-scale variance, the production term due to the eddy viscosity is dominant among positive terms. It was shown that the Reynolds stress, the eddy viscosity, and the length-scale variance are closely related to each other through the production terms in their transport equations. We compared the exact transport equation with the Spalart-Allmaras model and the three-equation model. We expect that the present analysis of the transport equation is helpful in deriving better turbulence models.

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