The mechanism of zero mean absolute vorticity state in rotating channel flow

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In the core region of spanwise rotating channel flows, the mean velocity profile is approximately linear with a slope of twice the system rotation rate. The mechanism of this zero mean absolute vorticity state is investigated from the turbulence modeling point of view. The mean velocity profile is calculated using three simple nonlinear eddy-viscosity models. It is shown that two models, the curvature-corrected type of explicit algebraic Reynolds stress model and the model with the corotational derivative of the second-order nonlinear term, reproduce well the zero mean absolute vorticity profile. Both models are derived by taking into account the effect of the advection of the Reynolds stress in the rotating frame. In particular, the latter model reflects the memory effect of the second-order nonlinear term. This effect means that a nonzero absolute vorticity creates a difference between the normal stresses, leading to a large shear stress in a rapidly rotating system. Since the actual value of the shear stress does not increase with the rotation rate, the absolute vorticity needs to be very small. To confirm the memory effect of the nonlinear term, anisotropic and temporally nonlocal effects of the mean velocity on the Reynolds stress are evaluated using Green’s function for the velocity fluctuation.

I. INTRODUCTION

The effect of system rotation on turbulence is very important in understanding and predicting various turbulent flows in nature and in engineering. As an example of rotating turbulence, spanwise rotating channel flow has been examined in experiment and in numerical simulation. It is well known that over a certain region of spanwise rotating channel flows, the mean velocity profile becomes approximately linear with a slope of twice the system rotation rate; that is, the mean absolute vorticity nearly vanishes. The zero mean absolute vorticity state can be seen in other rotating flows such as the Couette flow and a simple shear flow modulated periodically. Yanase and Kaga carried out the DNS of a minimal channel flow at low Reynolds number. They examined longitudinal vortical structures to clarify the mechanism of the zero mean absolute vorticity state. Using the Lie-group analysis, Oberlack theoretically obtained a solution for the mean velocity linear in the core region of the channel. Nevertheless, the mechanism has not been entirely understood especially for fully developed turbulent flows.

Rotating channel flow has also been simulated using the Reynolds-averaged Navier-Stokes equation models. The turbulence model relates the Reynolds stress to the mean velocity explicitly or implicitly. To accurately predict the zero mean absolute vorticity state, its mechanism needs to be expressed in model equations in some way. In the Reynolds stress equation model, the effect of system rotation on turbulence can be expressed explicitly as the Coriolis terms. Linear and nonlinear eddy-viscosity models are also used to predict rotating channel flows by incorporating the rotational effect. Although some of the simulations successfully predicted the zero mean absolute vorticity profile, the turbulence models using many model constants are too complex to provide a clear understanding of the profile.

In this work, we try to clarify the mechanism from the turbulence modeling point of view. A turbulence model is of course a tool for predicting practical turbulent flows. At the same time, it can be used to better understand the mechanism of turbulent flow. For example, the logarithmic law of the mean velocity in the turbulent boundary layer can be explained as follows. The turbulent shear stress can be expressed as the eddy-viscosity approximation

\[ \langle u'v' \rangle = -\nu_T \frac{\partial U}{\partial y}, \]  

where \( \nu_T \) is the eddy viscosity and \( \partial U/\partial y \) is the mean velocity gradient. It is expected that in a certain region of the turbulent boundary layer where the shear stress is constant, the turbulent length scale is proportional to the distance \( y \) from the wall. If \( \nu_T \) is proportional to \( y \) in the constant shear-stress region, the velocity gradient is proportional to \( 1/y \), resulting in the logarithmic velocity profile. In order to actually predict the mean velocity profile, it is necessary to model \( \nu_T \) in a specific form. Nevertheless, the eddy-viscosity representation (1) is useful in obtaining a qualitative understanding of the logarithmic law. The zero mean absolute velocity state is also a fundamental property of turbulent flow. We expect that it can be qualitatively understood using the nonlinear eddy-viscosity model.

Even if a turbulence model predicts a good velocity profile, it is possible that the model does not reflect the true mechanism of the turbulent flow. To verify the model expression, it is useful to evaluate terms in the transport equation...
for the Reynolds stress using DNS data. In particular, the production and Coriolis terms are closely related to the nonlinear eddy-viscosity model. However, it is also necessary to model the pressure-strain term to evaluate the direct relation of the Reynolds stress to the mean velocity gradient. In order to assess their relation without the help of modeling, we introduced Green’s function for the velocity fluctuation to derive an exact nonlocal expression for the Reynolds stress.20–22 Using the nonlocal expression, we evaluate anisotropic and temporally nonlocal effects of the mean velocity on the Reynolds stress to confirm the mechanism explained by the nonlinear eddy-viscosity model.

This paper is organized as follows. In the following section, linear and nonlinear eddy-viscosity models are examined for the rotating channel flow. The mean velocity profile is calculated using two simple nonlinear eddy-viscosity models. In Sec. III, the effect of the advection term is examined in the transport equation for the Reynolds stress. A nonlinear eddy-viscosity model with the corotational derivative of the second-order nonlinear term is used to explain the mechanism of the zero mean absolute vorticity state. In Sec. IV, to confirm the mechanism, anisotropic and temporally nonlocal effects of the mean velocity on the Reynolds stress are evaluated using Green’s function for the velocity fluctuation. Concluding remarks are given in Sec. V.

II. NONLINEAR EDDY-VISCOSITY MODEL

In order to provide data for later discussion, we carry out a DNS of a rotating channel flow to evaluate the mean velocity, the Reynolds stress, its transport equation, and Green’s function for the velocity fluctuation. We consider a plane channel rotating about a spanwise axis at the rate $\Omega$. In the reference frame rotating with the channel, the equations for the velocity $u_i$ are written as

$$\frac{\partial u_i}{\partial t} = -\frac{\partial}{\partial x_j} u_i u_j - \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} - 2\epsilon_{ijk}\Omega j u_k + \delta_{i1},$$

(2)

$$\frac{\partial u_i}{\partial x_i} = 0,$$

(3)

where $\Omega=(0,0,\Omega)$, $p$ is the pressure, $\nu$ is the molecular viscosity, and the summation convention is used for repeated indices. Variables $x_1(=x)$, $x_2(=y)$, and $x_3(=z)$ denote the coordinates in the streamwise, wall-normal, and spanwise directions, respectively; corresponding velocity components are given by $u_1(=u)$, $u_2(=v)$, and $u_3(=w)$. All quantities are normalized by the channel half-width $h$ and the wall-friction velocity $u_\tau [=(h\partial P/\partial x)^{1/2}]$, where $dP/\partial x$ is the external pressure gradient.

We carried out two runs with different rotation rates $Ro_i(=2\Omega h/u_\tau)=2.5$ and 7.5. Statistics for mainly the latter case will be examined. The Reynolds number is set to $Re_z(=u_\tau h/\nu)=150$. The size of the computational domain is $L_x \times L_y \times L_z = 12.8 \times 2 \times 3.2$ and the number of grid points is $N_x \times N_y \times N_z = 256 \times 96 \times 128$. Periodic boundary conditions for $u_i$ are used in the $x$ and $z$ directions, whereas no-slip conditions $u_i=0$ are imposed at the walls ($y=\pm h$). The computation was run for a sufficiently long time in order to be statistically independent of the initial condition. Then, statistics such as the Reynolds stress were accumulated over a time period of $10h/u_\tau$.

Figure 1 shows the mean velocity profiles obtained from the DNS for $Ro_i=2.5$ and 7.5. DNS result of Nishimura and Kasagi23 for $Ro_i=2.5$ is also plotted. The profiles of the shear stress for $Ro_i=2.5$ and 7.5 are nearly the same at $-0.8<y<0.3$, although the velocity gradient in the latter case is three times as large as that in the former case. These profiles mean that the eddy viscosity for $Ro_i=7.5$ is fairly smaller than that for $Ro_i=2.5$. The fact that the shear stress does not increase with the rotation rate will be mentioned again later.

From now on, using the turbulence model, we discuss the relation between the Reynolds stress and the mean velocity in the case of $Ro_i=7.5$ for which the rotational effect is more relevant. The objective is not to develop a nonlinear eddy-viscosity model that accurately predicts the mean velocity profile in the whole region of the rotating channel flow. We try to find a simple model that reproduces the zero mean absolute vorticity profile in the core region and to bet-

![Figure 1](image1.png)  
**FIG. 1.** Profiles of mean velocity obtained from DNS as a function of $y$ for $Ro_i=2.5$ and 7.5. DNS data of Nishimura and Kasagi (Ref. 23) are also plotted.

![Figure 2](image2.png)  
**FIG. 2.** Profiles of turbulent shear stress $\langle u'u' \rangle$ obtained from DNS as a function of $y$ for $Ro_i=2.5$ and 7.5. DNS result of Nishimura and Kasagi (Ref. 23) for $Ro_i=2.5$ is also plotted.
ter understand its mechanism. A typical nonlinear eddy-viscosity model in the rotating frame can be written as $^{24,25}$

$$\langle u'_i u'_j \rangle^* = -2 \nu_T S_{ij} - \eta_i (S_{ik} \bar{W}_k - \bar{W}_{ik} S_k) - \eta_j (S_{jk} \bar{W}_k - \bar{W}_{jk} S_k)^*$$

$$- \eta_i (\bar{W}_i \bar{W}_j)^* - \eta_j (\bar{W}_j \bar{W}_i)^* - \eta_k (\bar{W}_k \bar{W}_i S_{mj} + S_{ik} \bar{W}_k \bar{W}_m)^*$$

where

$$S_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right),$$

$$\bar{W}_{ij} = W_{ij} + \Omega_{ij}, \quad \bar{W}_i = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i} \right), \quad \Omega_{ij} = \epsilon_{ijk} \Omega_k.$$  

(4)

Here, the asterisk indicates the deviatoric part of a tensor $(a^*_{ij} = a_{ij} - a_{kk} \delta_{ij}/3)$, $\langle \rangle$ denotes ensemble averaging, $U_i(= (u_i))$ is the mean velocity, $u'_i$ is the velocity fluctuation, and $\eta_i (i = 1 – 5)$ are coefficients for the nonlinear terms. The vorticity tensor $W_{ij}$ appearing in the nonlinear eddy-viscosity model in the inertial frame is replaced by the absolute vorticity tensor $\bar{W}_{ij}$ in Eq. (4). $^{24,26–28}$ The first term on the right-hand side of Eq. (4) is the linear eddy viscosity term whereas the second to fourth terms and the fifth to sixth terms are second- and third-order nonlinear terms, respectively. Higher-order terms can also be included, but they are omitted here for simplicity. Coefficients $\nu_T$ and $\eta_j$ should be expressed in terms of turbulent quantities. In the $k$-$\varepsilon$ model, the turbulent kinetic energy $k$ and its dissipation rate $\varepsilon$ are used to express the coefficients; other scalars such as $S_{ij}$ and $\bar{W}_{ij}$ can also be used. For the spanwise rotating channel flow, nonzero components of the strain and vorticity tensors are given by

$$S_{12} = S_{21} = \frac{1}{2} \frac{\partial U}{\partial y}, \quad \bar{W}_{12} = - \bar{W}_{21} = \frac{1}{2} \frac{\partial U}{\partial y} - \Omega,$$

$$\Omega_{12} = - \Omega_{21} = - \Omega.$$  

(7)

The explicit algebraic Reynolds stress model (EARSM) is one of the nonlinear eddy-viscosity models expressed in Eq. (4). Moreover, the EARSM was modified by taking into account the effects of the system rotation and the streamline curvature. $^{16,29}$ In this curvature-corrected EARSM, instead of $\bar{W}_{ij}$ the following vorticity tensor is used:

$$W_{ij} = \bar{W}_{ij} + C_\Omega \Omega_{ij},$$

(8)

where $C_\Omega$ is a nondimensional model constant. The derivation of this generalized vorticity tensor will be described in the next section.

First, we discuss the rotational effect in the framework of the linear eddy-viscosity approximation. In the standard $k$-$\varepsilon$ model, the shear stress is expressed as

$$\langle u' v' \rangle = -2 \nu_T S_{12}, \quad \nu_T = C_\nu \frac{k^2}{\varepsilon}. \quad (9)$$

where $C_\nu$ is a model constant. It is known that the standard $k$-$\varepsilon$ model using the linear eddy viscosity can predict only a symmetric velocity profile because the system rotation rate $\Omega$ does not appear in the $k$ and $\varepsilon$ equations. $^{12}$ In order to properly predict an asymmetric velocity profile, one can add some terms representing the rotational effect to the $\varepsilon$ equation. Howard et al. $^{12}$ proposed the following expression for the Bradshaw-Richardson number:

$$B = -2 \Omega \left( \frac{\partial U}{\partial y} - 2 \Omega \right) \frac{k^2}{\varepsilon^2},$$

(10)

and added it to the coefficient of the destruction term in the $\varepsilon$ equation. The Bradshaw-Richardson number is a criterion for the stability due to the rotational effect; that is, the flow is expected to be stable (unstable) for a positive (negative) value of $B$. Using a statistical theory called the two-scale direct interaction approximation (TSDIA), Shimomura $^{31}$ proposed an additional term proportional to $k \Omega \cdot \nabla \times \bar{u}$ in the $\varepsilon$ equation. This term was adopted in the simulations of Wizman et al. $^{11}$ and of Nagano and Hattori. $^{17}$ However, it is very difficult to theoretically model and verify the $\varepsilon$ equation. $^{32}$ Instead of modifying the $\varepsilon$ equation, which affects the Reynolds stress indirectly through the eddy viscosity, we consider the explicit effect of the system rotation on the Reynolds stress.

The rotational effect can be directly incorporated into the eddy viscosity $\nu_T$ in the linear eddy-viscosity model. For example, by mimicking the behavior of the Reynolds stress equation model in rotating homogeneous shear flows, Pettersson Reif et al. $^{15}$ proposed a modified coefficient for the eddy viscosity in the $k$-$\varepsilon$-$\nu^2$ model as

$$\nu_T = C'_\mu \langle \bar{u}' \bar{v}' \rangle^2 \tau,$$

(11)

$$C'_\mu = C_\mu \frac{1 + \alpha_2 |\xi_1| + \alpha_3 |\xi_3|}{1 + \alpha_4 |\xi_1|} \times \left( \frac{1 + \alpha_1 \xi_3}{1 + \alpha_2 \xi_2} + \alpha_3 \sqrt{|\xi_2| - |\xi_3|} \right)^{-1},$$

(12)

where

$$\xi_1 = \tau^2 S_{ij} S_{ij}, \quad \xi_2 = \tau^2 W_{ij} W_{ij}, \quad \xi_3 = \xi_1 - \xi_2.$$  

(13)

Here, $C_\mu$ and $\alpha_i (i = 1–5)$ are nondimensional model constants and $\tau$ is the turbulent time scale. Although good velocity profiles of rotating channel flows were predicted, the above expression is too complex to gain insight into the mechanism of the zero mean absolute vorticity state. It seems too restrictive to modify the eddy viscosity only.

Next, we examine the zero mean absolute vorticity state using the nonlinear eddy-viscosity model given by Eq. (4). For the rotating channel flow the shear stress can be expressed as

$$\langle u' v' \rangle = -2 \nu_T S_{12} + 2 \eta_5 \bar{W}_{12} S_{12}. \quad (14)$$

The difference between the normal components is given by
\[ \langle u'^2 \rangle - \langle v'^2 \rangle = 4 \eta_1 S_{12} \overline{W}_{12} \],
\]
which plays an important role in the development of \( \langle u' v' \rangle \) as will be discussed later. The shear stress is expressed in terms of the first- and third-order nonlinear terms whereas the normal components are expressed in terms of the second-order nonlinear term. It is known that third-order terms are necessary to properly predict the mean swirl velocity in an axially rotating pipe.\[^{33}\] We expect that the third-order term is also useful for reproducing the zero mean absolute vorticity profile in the rotating channel flow. Since the second term on the right-hand side of Eq. (14) is proportional to \( S_{12} \), the effective eddy viscosity is given by \( \nu_T = \eta_2 \overline{W}_{12} \). If the value of \( \overline{W}_{12} \) is very large, this eddy viscosity may have a negative value, causing a numerical instability. In order to avoid the instability we make use of the Padé approximation\[^{13}\] to derive the following expression:
\[ \langle u' v' \rangle_1 = \frac{-2 \nu_T S_{12}}{1 + C_{\nu f} f_c \frac{k^2}{\epsilon^2} \overline{W}_{12}^2}, \tag{16} \]
where
\[ \nu_T = C_{\nu f} f_c \frac{k^2}{\epsilon^2}. \tag{17} \]

Here, \( C_{\nu f} \) is a model constant and \( f_c \) is the wall-damping function. Using the \( k-\epsilon \) model with the above expression, we calculate the mean velocity profile of the rotating channel flow at \( Re_\theta = 150 \) to demonstrate the role of the third-order term. The standard equations for \( k \) and \( \epsilon \) are solved and a low-Reynolds-number model by Abe et al.\[^{34}\] is used for \( f_c \).

In addition, the curvature-corrected model is also solved in the same form
\[ \langle u' v' \rangle_2 = \frac{-2 \nu_T S_{12}}{1 + C_{\nu f} f_c \frac{k^2}{\epsilon^2} \overline{W}_{12}^2}, \tag{18} \]
where \( \overline{W}_{12} \) in Eq. (16) is replaced by \( \overline{W}_{12}^2 \). We should note that in the original EARSM, the expressions for \( \nu_T \) and \( \eta_2 \) are more complicated than Eqs. (17) and (18) because they include several invariants such as \( S_{ij} \) and \( \overline{W}_{ij}^2 \). The complicated coefficients are derived by solving the Reynolds stress equation exactly, and they are necessary to accurately predict the mean velocity profile in the whole region. In the present analysis, however, we adopt model expressions as simple as possible to better understand the role of nonlinear terms in predicting the zero mean absolute vorticity state.

Figure 3 shows the mean velocity profiles obtained from the \( k-\epsilon \) model with expressions \( \langle u' v' \rangle_1 \) and \( \langle u' v' \rangle_2 \) for \( Re_\theta = 7.5 \). Two results with \( C_{\nu f} = 0.15 \) and 0.2 are plotted for each case. The value of \( C_{\Omega} \) in Eq. (8) is set to 1.39 according to the curvature-corrected EARSM of Wallin and Johansson.\[^{16}\] In the case of \( \langle u' v' \rangle_1 \), the velocity gradient in the core region is fairly close to \( 2\Omega \). For \( \langle u' v' \rangle_2 \) to be different from the linear eddy-viscosity model and to predict an asymmetric velocity profile, the value of the parentheses in Eq. (16) needs to be greater than unity. This condition requires that the value of \( \overline{W}_{12}^2 \) should be nonzero and the velocity gradient deviates from \( 2\Omega \). A less steep velocity gradient of the rotating channel flow like the profiles of \( \langle u' v' \rangle_1 \) in Fig. 3 was also reported in the simulation of Nagano and Hattori\[^{17}\] that uses \( \overline{W}_{ij} \) as the vorticity tensor.

On the other hand, the velocity gradient in the core region in the case of \( \langle u' v' \rangle_2 \) is much closer to \( 2\Omega \). This agreement means that the following condition approximately holds:
\[ \overline{W}_{12} \left( \frac{1}{2} \frac{\partial U}{\partial y} - (1 + \Omega) \Omega \right) = -C_{\Omega} \Omega. \tag{19} \]
The good prediction by the curvature-corrected EARSM with \( \overline{W}_{ij} \) in contrast to the traditional EARSM with \( \overline{W}_i \) was also reported in Wallin and Johansson\[^{16}\] and Gatski and Wallin.\[^{29}\] The curvature-corrected model can well reproduce the zero mean absolute vorticity state. Nevertheless, it is not yet clear why the relation (19) holds. Since it includes the model constant \( C_{\Omega} \), the agreement with the zero mean absolute vorticity state depends on the value of the constant.

III. EFFECT OF ADVECTION OF THE REYNOLDS STRESS

To explain why the curvature-corrected model gives good velocity profiles and to further investigate the mechanism of the zero mean absolute vorticity state, we consider the transport equation for the Reynolds stress \( R_{ij} = \langle u_i' u_j' \rangle \) given by
\[ \frac{\partial R_{ij}}{\partial t} = P_{ij}^{(1)} + P_{ij}^{(2)} + P_{ij}^{(3)} + \Pi_{ij} - \epsilon_{ij} + D_{ij}, \tag{20} \]
where
\[ \frac{\partial R_{ij}}{\partial t} = \frac{\partial R_{ij}}{\partial t} + \Omega_{ik} R_{kj} - R_{ik} \Omega_{kj}, \tag{21} \]
\[ P_{ij}^{(1)} = -\frac{4}{3} k S_{ij}, \quad P_{ij}^{(2)} = -(R_{ik} S_{kj} + S_{ik} R_{kj}), \tag{22} \]
\[ P_{ij}^{(3)} = R_{ik} \overline{W}_{kj} - \overline{W}_{ik} R_{kj}. \]
\[ \Pi_{ij} = \left\langle p \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right\rangle, \quad \varepsilon_{ij} = 2 \nu \left\langle \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} \right\rangle, \quad (23) \]

\[ D_{ij} = -\frac{\partial}{\partial x_k} \left( \langle u_i' u_j' \rangle + \langle p' u_i' \rangle \delta_{jk} + \langle p' u_j' \rangle \delta_{ik} \right) \frac{\partial R_{ij}}{\partial x_k}, \quad (24) \]

and \( D/\partial t = \partial/\partial t + U \partial/\partial x_i \). The production term is divided into three parts in Eq. (22). On the left-hand side of Eq. (20), the corotational derivative \( \partial R_{ij}/\partial t \) is used instead of the material derivative \( DR_{ij}/\partial t \). As explained in the Appendix, the corotational derivative properly represents the unsteady behavior of a tensor in the rotating frame. For the rotating channel flow \( DR_{ij}/\partial t \) vanishes but \( D\bar{R}_{ij}/\partial t \) does not.

The transport equation for the deviatoric part of the Reynolds stress is given by

\[ \frac{D}{\partial t} \left( \begin{array}{c} R_{ij}^* \\ \bar{R}_{ik} \end{array} \right) = 0, \quad (26) \]

\[ D_{ij}^* - \frac{\bar{R}_{ik} \bar{D}_{ik}}{2k} = 0. \quad (27) \]

The pressure-strain and dissipation terms can be modeled as

\[ \Pi_{ij} - \varepsilon_{ij} = -R_{ij}^*/\tau + C'_3 k S_{ij} + C'_2 (R_{ik}^* S_{kj} + \bar{S}_{ik} \bar{R}_{kj})^* + C'_1 (R_{ik}^* \bar{W}_{kj} - \bar{W}_{ik} \bar{R}_{kj}), \quad (28) \]

where \( \tau \) is the turbulent time scale proportional to \( k/\epsilon \) and \( C'_i (i=1-3) \) are model constants. Substituting Eqs. (26)–(28) into Eq. (25) we obtain an algebraic Reynolds stress model (ARSM) as follows:

\[ R_{ij}^* = -C_1 k \tau S_{ij} - C_2 \tau (R_{ik}^* S_{kj} + \bar{S}_{ik} \bar{R}_{kj})^* + C_3 \tau (R_{ik}^* \bar{W}_{kj} - \bar{W}_{ik} \bar{R}_{kj}), \quad (29) \]

where \( C_1 = \frac{4}{3} - C'_1, \quad C_2 = 1 - C'_2, \quad \text{and} \quad C_3 = 1 + C'_3. \) Although the original ARSM \(C_0\) includes the ratio of the energy production to dissipation \( P_k/\epsilon \), the ratio is assumed to be unity here for simplicity. The EARS represents the explicit solution of Eq. (29) obtained from linear algebra using integrity bases. A non-linear eddy-viscosity model can also be derived by solving Eq. (29) iteratively. The first term on the right-hand side of Eq. (29) corresponds to the linear eddy-viscosity term where as the second and third terms give nonlinear eddy-viscosity terms.

In the ARSM given by Eq. (29), the advection term \( D\bar{R}_{ij}/\partial t \) is neglected. To examine whether the advection term is important, we evaluate it as well as the production and pressure-strain terms in the transport equation for \( \langle u' u' \rangle \) in the DNS. Figure 4 shows profiles of the following terms:

\[ \bar{D}_{12} = \langle \langle u'^2 \rangle - \langle v'^2 \rangle \rangle \Omega, \quad (30) \]

\[ P_{12} = -\frac{4}{3} k S_{12}, \quad P_{21} = -\left( \langle u'^2 \rangle + \langle v'^2 \rangle - \frac{4}{3} k \right) S_{12}, \quad (31) \]

\[ P_{12}^* = \left( \langle u'^2 \rangle - \langle v'^2 \rangle \right) \bar{W}_{12}, \quad (32) \]

obtained from the DNS for \( R_{0}=7.5 \). At \(-0.8 < y < 0.3 \), where the mean absolute vorticity nearly vanishes, the advection term \( \bar{D}_{12}/\partial t \), the production term \( P_{12}(1) \), and the pressure-strain term \( \Pi_{12} \) are dominant. This result means that the effect of the advection term is important and that the weak-equilibrium condition (26) for the traditional ARSM is not justified for the rotating channel flow. This is why the model expression (16) with \( \bar{W}_{12} \) gives inaccurate velocity profiles in Fig. 3.

In the curvature-corrected ARSM, the effect of the advection term is included as follows. Since in the rotating channel flow the material derivative \( DR_{ij}/\partial t \) vanishes, instead of Eq. (26), the following condition holds exactly:

\[ \frac{D}{\partial t} \left( \begin{array}{c} R_{ij}^* \\ \bar{R}_{ik} \end{array} \right) = -\frac{1}{k} (R_{ik}^* \Omega_{kj} - \Omega_{ik} \bar{R}_{kj}). \quad (33) \]

If Eq. (33) is substituted into Eq. (25), the right-hand side of Eq. (33) can be incorporated into the production term \( P_{ij}^* \), leading to

\[ R_{ij}^* = -C_1 k \tau S_{ij} - C_2 \tau (R_{ik}^* S_{kj} + \bar{S}_{ik} \bar{R}_{kj})^* + C_3 \tau (R_{ik}^* \bar{W}_{kj} - \bar{W}_{ik} \bar{R}_{kj}), \quad (34) \]

where \( \bar{W}_{ij} \) is given by Eq. (8) and \( \Omega_0 = 1/C_3 = 1/(1+C_3') \). Since the effect of the advection term is taken into account, the curvature-corrected model (18) gives good result in Fig. 3. The model constant \( C_0 \) is related to \( C_3 \) appearing in the model for \( \Pi_{ij} - \varepsilon_{ij} \) in Eq. (28). Therefore, whether the velocity gradient is close to \( 2 \Omega \) or not depends on the model constant \( C_3 \); the zero mean absolute vorticity state is obtained only when the value of \( C_3 \) is appropriate.

Next, we consider the effect of the advection term in a different way. Although the condition (33) is exact for the
rotating channel flow, it does not always hold for general rotating flows. Instead of assuming the zero material derivative, we try to treat the advection term explicitly as follows:

\[ R_{ij}^* = -C_1k\tau_{ij} + 2C_1C_2^2/3\tau^2(S_{ik}S_{kj})^* - C_1C_3^2k^2/3(S_{ik}\tilde{W}_{kj})^* \]

\[ + \tilde{W}_{ik}(S_{kj}) - \tau \frac{D\tilde{R}_{ij}^*}{Dt}. \]  

(35)

Since this treatment prevents us from obtaining an explicit solution using integrity bases, we need to solve the equation iteratively. By substituting the expression iteratively up to the second order of the strain and vorticity tensors and the first order of the time derivative, we obtain

\[ R_{ij}^* = -C_1k\tau_{ij} + 2C_1C_2^2/3\tau^2(S_{ik}S_{kj})^* - C_1C_3^2k^2/3(S_{ik}\tilde{W}_{kj})^* - C_1C_3^2k^2/3(S_{ik}\tilde{W}_{kj} - \tilde{W}_{ik}S_{kj})]. \]

(36)

For the rotating channel flow, the shear stress is expressed as

\[ \langle u'v' \rangle = -C_1k\tau_{12} - 4C_1C_3^2k^2\Omega S_{12}\tilde{W}_{12}. \]

(37)

When \( C_1C_3\tilde{W}_{12} \) is negative, the effective eddy viscosity may have a negative value. To avoid the numerical instability we also treat the following form:

\[ \langle u'v' \rangle = -C_1k\tau_{12} + 4C_1C_3^2\tau^2\Omega S_{12}\tilde{W}_{12}, \]

(38)

where the Reynolds stress is restored in the second term on the right-hand side. Combining Eqs. (37) and (38), we consider the following expression for the \( k-e \) model

\[ \langle u'v' \rangle_3 = -2

C_4\tilde{W}_{12}g_{12} \]

(39)

\[ g_{12} = \begin{cases} 
1 + C_{43}k^2/\varepsilon \Omega \tilde{W}_{12} & \text{for } \tilde{W}_{12} \geq 0, \\
1 - C_{43}k^2/\varepsilon \Omega \tilde{W}_{12}^{-1} & \text{for } \tilde{W}_{12} < 0,
\end{cases} \]

where \( \nu_1 \) is given by Eq. (17) and \( C_{43} \) is a model constant. The expression for \( g_{12} \) for \( \tilde{W}_{12} < 0 \) is the Padé approximation to that for \( \tilde{W}_{12} \geq 0 \).

Figure 5 shows the mean velocity profiles obtained from the \( k-e \) model with expression \( \langle u'v' \rangle_3 \) for \( \text{Ro}_o=7.5 \). Three results with \( C_{43}=0.5, 1, \) and 2 are plotted. At \(-0.8 < y < 0.3\), the velocity gradient is close to \( 2\Omega \), especially in the case of \( C_{43}=2 \). This good agreement can be explained as follows. In contrast to Eqs. (16) and (18), the model expression (39) explicitly includes \( \Omega \) originated from the corotational derivative. In a rapidly rotating system, the coefficient \( C_3 = (k^2/\varepsilon)\Omega \) can have a large value, leading to a large value of the shear stress. As shown in Fig. 2, the actual value of the shear stress does not increase with the rotation rate. This fact requires that \( \tilde{W}_{12} \) should be very small and it tends to zero for a larger value of \( C_{43} \). We should note that the model expression (39) does not include \( C_1 \). The zero mean absolute vorticity state is obtained independently of the modeling of \( \Pi_{ij} = e_{ij}^* \).

The model expressions examined were derived under several assumptions. For example, the ratio \( P_k/e \) was assumed to be unity in deriving Eq. (29). Here, we briefly examine the influence of the value of \( P_k/e \) on the solution. If the value of \( P_k/e \) is not equal to unity but is still assumed to be constant, the model expression (39) can be rewritten as

\[ \langle u'v' \rangle_4 = -2\nu_2 S_{12}g_{12} \]

(40)

\[ g_{12} = \begin{cases} 
-f_{P_k} \left( 1 + C_{43}f_{P_k} k^2/\varepsilon \Omega \tilde{W}_{12} \right) & \text{for } \tilde{W}_{12} \geq 0, \\
-f_{P_k} \left( 1 - C_{43}f_{P_k} k^2/\varepsilon \Omega \tilde{W}_{12}^{-1} \right) & \text{for } \tilde{W}_{12} < 0,
\end{cases} \]

where \( f_{P_k} = 1/(1 + C_{43}[(P_k/e) - 1]) \) and \( C_{43} \) is a model constant. Figure 6 shows the mean velocity profiles obtained from the \( k-e \) model with expression \( \langle u'v' \rangle_4 \) given by Eq. (40) for \( \text{Ro}_o=7.5 \). The constants are set to \( C_{43}=2 \) and \( C_{44}=1/1.8 \). Three results with \( P_k/e=0.5, 1, \) and 2 are plotted. The two profiles for \( P_k/e=0.5 \) and 1 are nearly the same. Although the profile for \( P_k/e=2 \) is slightly overestimated in the core region compared to the former two profiles, the velocity gradient is nearly the same. Therefore, the result that the model expression with the corotational derivative can reproduce the zero mean absolute vorticity profile is obtained irrespective of the value of \( P_k/e \).

The model expression (37) can be written in a tensorial form as
\[ R'_{ij} = -2\nu_S S_{ij} + \tau \frac{D}{Dt}[\eta(S_{ik}\bar{W}_{kj} - \bar{W}_{ik}S_{kj})]. \]  

(41)

The time derivative is not commonly used in nonlinear eddy-viscosity models. As an example, Speziale adopted the Oldroyd derivative defined as

\[ \frac{DS_{ij}}{Dt} = \frac{DS_{ij}}{dt} - \frac{\partial U_i}{\partial x_k} S_{kj} - \frac{\partial U_j}{\partial x_k} S_{ki}. \]  

(42)

Using the TSDIA, Yoshizawa also proposed a nonlinear eddy-viscosity model using \( D_S/\bar{D}_t \). In order to take into account the streamline curvature effect in the algebraic stress model, Girimaji made use of the acceleration vector \( d\bar{U}/dt \) to determine the appropriate coordinate system where the weak-equilibrium condition is invoked. Although they seem somewhat unusual compared to traditional eddy-viscosity models, we believe that the time derivative terms can be candidates for quantities representing unsteady and streamline-curve effects on turbulent flows.

The role of the corotational derivative term can be discussed in connection with the Bradshaw-Richardson number mentioned in the preceding section. The model expression (39) is closely related to the Bradshaw-Richardson number \( \nu_S = 1 + C_{v3}\bar{W}_{12} = 1 - \frac{1}{4} C_{v3}B \).

(47)

The stability criterion can be explained by the production term for the turbulent energy. The production term is written as

\[ P_k = -R'_{ij} S_{ij} = 2\nu_S S_{ij} \left( 1 - \frac{1}{4} C_{v3}B \right). \]  

(48)

Therefore, when \( B \) is positive (negative), the energy production decreases (increases) because of the rotational effect.

The Bradshaw-Richardson number given by Eq. (10) is defined only for the unidirectional flow such as a channel flow. Since Eq. (10) cannot be adopted for a three-dimensional flow, Spalart and Shur proposed a scalar measure of rotation and curvature effects given by

\[ B = \bar{W}_{ik} S_{jk} \frac{D S_{ij}}{Dt} \frac{k^2}{\bar{D}^2 S_{mn}}. \]  

(49)

This quantity can be considered as a generalized Bradshaw-Richardson number. We can show that the above expression for \( B \) is obtained from the tensorial form (41) of the present model. Since the corotational derivative of the absolute vorticity, \( D\bar{W}_{ik}/Dt \), vanishes for the rotating channel flow, we assume that \( D\bar{W}_{ik}/Dt(\eta S_{ik}\bar{W}_{kj}) = \eta(\bar{D}S_{ik}/Dt)\bar{W}_{kj} \).

The model expression (41) can be written as

\[ R'_{ij} = -2\nu_S S_{ij} + \tau \eta \left( \frac{\bar{D}S_{ik}}{Dt}\bar{W}_{kj} - \frac{\overline{D}S_{ik}}{Dt} \right). \]  

(50)

The energy production term is then given by

\[ P_k = 2\nu_S S_{ij} \bar{W}_{ij} - 2\eta \frac{\bar{D}S_{ik}}{Dt}\bar{W}_{kj} S_{ij} = 2\nu_S S_{ij}(1 - C_{v3}B), \]  

(51)

where \( B \) is given by Eq. (49) and \( C_{v3} \) is a nondimensional constant. Therefore, the model expression (41) can also account for the generalized Bradshaw-Richardson number for three-dimensional flows. This fact suggests that the corotational derivative term properly represents the rotational effect.
IV. ANALYSIS USING GREEN'S FUNCTION

To confirm the interpretation of the model expression (41) we evaluate statistics in the DNS in more detail. As shown in Fig. 4, balance of the transport equation is useful for understanding the mechanism of the Reynolds stress production. However, in order to know the quantitative dependence of the Reynolds stress on the mean strain and vorticity tensors, the pressure-strain term needs to be modeled and some ambiguity remains. Hamba used Green’s function for the velocity fluctuation to derive an exact nonlocal expression for the Reynolds stress. In this section, using the nonlocal expression, we evaluate the direct effect of the mean velocity on the Reynolds stress without the help of modeling.

The equations for the velocity fluctuation in the rotating frame are given by

\[
\frac{Du'_i}{Dt} + \Omega_j u'_j + \frac{\partial}{\partial x_j} (u'_j u'_i - \langle u'_j u'_i \rangle) + \frac{\partial p'_i}{\partial x_j} - \nu \frac{\partial^2 u'_i}{\partial x_j \partial x_j} = -u'_j (S_{ij} + \bar{W}_{ij}),
\]

\[
\frac{\partial u'_i}{\partial x_i} = 0,
\]

We then introduce Green’s function \(g_{ij}(x,t;x',t')\), corresponding to the left-hand side of Eq. (52); its equations are given by

\[
\frac{Dg_{ij}}{Dt} + \Omega_k g_{kj} + \frac{\partial}{\partial x_k} (u_k g_{ij} - \langle u_k g_{ij} \rangle) + \frac{\partial g_{ij}}{\partial x_j} - \nu \frac{\partial g_{ij}}{\partial x_k} = \delta_{ij} \delta(x-x') \delta(t-t'),
\]

\[
\frac{\partial g_{ij}}{\partial x_j} = 0,
\]

where \(p_{ij}\) is a vector that plays a similar role to the pressure and guarantees the solenoidal condition (55) for \(g_{ij}\). Using Green’s function, a formal solution for the velocity fluctuation can be written as

\[
u_t'(x,t) = - \int dx' \int_0^t dt' g_{ij}(x,t;x',t') u'_j(x',t') \\
\times [S_{jk}(x',t') + \bar{W}_{jk}(x',t')].
\]

The solution (56) leads to a nonlocal expression for the Reynolds stress given by

\[
\langle u'_i u'_j \rangle(x,t) = - \int dx' \int_0^t dt' \nu_{NLijkm}(x,t;x',t') \\
\times [S_{km}(x',t') + \bar{W}_{km}(x',t')],
\]

where

\[
\nu_{NLijkm}(x,t;x',t') = \left[ \langle u'_i(x,t) g_{jk}(x,t;x',t') u'_m(x',t') \rangle \right] + \left[ \langle u'_j(x,t) g_{ik}(x,t;x',t') u'_m(x',t') \rangle \right] / 2.
\]

Therefore, the Reynolds stress can be expressed as a space and time integral of the mean strain and vorticity tensors.

Using this expression, we examine whether the memory effect of the second-order nonlinear term is important. Equation (57) involves not only the time integral like Eq. (44) but also the spatial integral. Since it is too complex to estimate both spatially and temporally nonlocal effects, we need to assume the local approximation in space. Before investigating the temporally nonlocal effect, we try to examine the validity of the local approximation in space as follows.

The nonlocal eddy viscosity \(\nu_{NLijkm}(x,t;x',t')\) is expected to have a nonzero value if the distance \(|x-x'\)| is less than the turbulent length scale. If the mean strain and vorticity tensors are nearly constant in the region, the Reynolds stress can be approximated as

\[
\langle u'_i u'_j \rangle_{SL}(x,t) = - \int_0^t dt' \nu_{SLijkm}(x,t;t') \\
\times [S_{km}(x,t') + \bar{W}_{km}(x,t')],
\]

where

\[
\nu_{SLijkm}(x,t;x',t') = \int dx' \nu_{NLijkm}(x,t;x',t').
\]

Figure 7 shows profiles of the turbulent shear stress for \(R_o = 7.5\). The solid line denotes the value obtained directly by averaging \(u'u'\), the dashed line stands for the nonlocal expression given by Eq. (57), and the dotted line denotes \(\langle u'u' \rangle_{SL}\) given by Eq. (59). The agreement between the directly obtained value and the value of Eq. (57) shows that the nonlocal expression is accurate. On the other hand, the value of \(\langle u'u' \rangle_{SL}\) is fairly different from the directly obtained value near the wall. This is because the mean strain and vorticity tensors rapidly change near the wall compared to those in the core region. This deviation does not necessarily mean that any local models give inaccurate results; some compensation can be made near the wall such as the wall-damping function. However, this result suggests the difficulty in modeling the near-wall region in a universal manner. The local approximation is relatively good in the core region compared to the near-wall region. In the present analysis, we pay attention to velocity profiles in the core region.

Now we examine the anisotropy of the eddy viscosity tensor; that is, the dependence of the shear stress on \(S_{ij}\) and...
\( \vec{W}_{ij} \)

By this analysis, we can see which nonlinear term in Eq. (4) contributes to the shear stress. We divide the velocity fluctuation into three parts as

\[
\begin{align*}
\vec{u}_i' &= \vec{u}_{S_i} + \vec{u}_{W_i} + \vec{u}_{\Omega_i},
\end{align*}
\]

where

\[
\begin{align*}
\vec{u}_{S_i}(x,t) &= -\int dx' \int_0^t dt' g_{ij}(x,t;x',t') \vec{u}_j(x',t') S_{\eta j}(x',t'), \\
\vec{u}_{W_i}(x,t) &= -\int dx' \int_0^t dt' g_{ij}(x,t;x',t') x \vec{u}_j(x',t') W_{\eta j}(x',t'), \\
\vec{u}_{\Omega_i}(x,t) &= -\int dx' \int_0^t dt' g_{ij}(x,t;x',t') x \vec{u}_j(x',t') \Omega_{\eta j}.
\end{align*}
\]

We pay attention not only to the difference between \( S_{ij} \) and \( \vec{W}_{ij} \), but also to the difference between \( \vec{W}_{ij} \) and \( \Omega_{ij} \). Let us define \( \eta_i' = 4C_1 C_2 \kappa^3 \Omega \) that appears in Eq. (37). If \( \eta_i' \) is large in a rapidly rotating system while \( \langle u' v' \rangle \) does not increase with \( \Omega \), the absolute vorticity \( \vec{W}_{12} \) needs to be very small, as discussed previously. Therefore, we need to know whether \( \eta_i' \) is actually large or not. However, if we treat only the sum of \( \vec{u}_{W_i} \) and \( \vec{u}_{\Omega_i} \), we will find that the term \( \eta_i' S_{12} W_{12} \) is small without knowing the magnitude of \( \eta_i' \) because \( \vec{W}_{12} \approx 0 \). By dividing the velocity into the three parts, we can distinguish between \( \eta_i' S_{12} W_{12} \) and \( \eta_i' S_{12} \Omega_{12} \) to estimate the magnitude of \( \eta_i' \).

Using the three components (62)–(64), we can divide the Reynolds shear stress into nine terms as follows:

\[
\langle u' v' \rangle = \langle u'_S v'_S \rangle + \langle u'_W v'_W \rangle + \langle u'_\Omega v'_\Omega \rangle + \langle u'_W v'_S \rangle + \langle u'_\Omega v'_W \rangle + \langle u'_S v'_W \rangle + \langle u'_W v'_\Omega \rangle + \langle u'_\Omega v'_S \rangle + \langle u'_S v'_\Omega \rangle.
\]

The second term on the right-hand side of Eq. (37) corresponds to the first parentheses on the right-hand side of Eq. (65). Figure 8 shows profiles of the nine terms and their total obtained from the DNS for \( R_{0r} = 7.5 \). The total term and the three large terms are plotted by symbols. Since the linear eddy-viscosity term is related to \( P^{(1)}_{12} = -(4/3) k S_{12} \) and the turbulent energy \( k \) itself is created by the mean shear \( S_{12} \), the term \( \langle u'_S v'_S \rangle \) represents mainly the linear eddy-viscosity part of \( \langle u' v' \rangle \). On the other hand, it is clearly seen that \( \langle u'_W v'_W \rangle \) and \( \langle u'_\Omega v'_\Omega \rangle \) are dominant in magnitude and are nearly balanced with respect to each other. These two terms correspond to \( \eta_1 S_{12} W_{12} \) and \( \eta_1 S_{12} \Omega_{12} \), respectively, and it is suggested that \( \eta_1 \) is large and the contribution of the second term on the right-hand side of Eq. (37) is important.

Next, we explain how the two dominant terms are created by the memory effect. Since the corotational derivative of the absolute vorticity, \( \vec{D} \vec{W}_{ij}/D(t) \), vanishes for the rotating channel flow, there is no unsteady effect of \( \vec{W}_{ij} \); the unsteady effect of \( S_{ij} \) is expected to be important. We examine the dependence of the Reynolds stress in Eq. (57) on \( S_{ij}(t') \) as follows. The velocity components \( u'_i \) and \( u'_S \) can be expressed as the time integrals given by

\[
\begin{align*}
u'_i(x,t) &= \int_0^t dt' \nu'_i(x,t;\tau'), \\
\nu'_S(x,t) &= \int_0^t dt' \nu'_{ST}(x,t;\tau'),
\end{align*}
\]

respectively, where

\[
\begin{align*}
u'_S(x,t;\tau') &= -\int dx' g_{ij}(x,t;x',\tau') u'_j(x',\tau') \\
&\times [S_{\eta j}(x',\tau') + \vec{W}_{\eta j}(x',\tau')], \\
\nu'_{ST}(x,t;\tau') &= -\int dx' g_{ij}(x,t;x',\tau') u'_j(x',\tau') S_{\eta j}(x',\tau').
\end{align*}
\]

Using \( v'_i \) and \( v'_{ST} \), we can express \( \langle u' v' \rangle \) and the three main terms as

\[
\begin{align*}
\langle u' v' \rangle &= \int dt' \langle u'(x,t) v'_i(x,t;\tau') \rangle, \\
\langle u'_W v'_W \rangle &= \int dt' \langle u'_W(x,t) v'_W(x,t;\tau') \rangle, \\
\langle u'_\Omega v'_\Omega \rangle &= \int dt' \langle u'_\Omega(x,t) v'_\Omega(x,t;\tau') \rangle.
\end{align*}
\]

The integrands appearing in Eqs. (70)–(73) are each a kind of two-time velocity correlation. For example, the integrand \( \langle u'_W(x,t) v'_W(x,t;\tau') \rangle \) in Eq. (72) represents the effect of \( S_{\eta j}(t') \) for \( t' < t \) on \( u'_W(x,t) \) at time \( t \). By evaluating the integrands, we can estimate the memory effect of each nonlinear term.

Figure 9 shows profiles of the integrands for the total and the three large terms obtained from the DNS as functions of \( t - t' \) at \( y = -0.18 \) for \( R_{0r} = 7.5 \). The half-period of \( \sin(2\Omega(t-t')) \) is also plotted for comparison. The magnitude of \( \langle u' v'_W \rangle \) and \( \langle u'_W v'_W \rangle \) decreases as \( t - t' \) increases. At
very small to keep the value of the shear stress. It was also shown that the model expression can account for the Bradshaw-Richardson numbers for two- and three-dimensional flows as the modification of the energy production term.

To confirm the memory effect of the nonlinear term, anisotropic and temporally nonlocal effects of the mean velocity on the Reynolds stress were evaluated using Green’s function. The second-order nonlinear terms involving the absolute vorticity tensor are dominant and their unsteady effect is clearly seen. The mechanism of the zero mean absolute vorticity state explained by the simple nonlinear eddy-viscosity model is verified by the analysis using Green’s function.

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APPENDIX: COROTATIONAL DERIVATIVE

In this appendix, we explain the meaning of the corotational derivative using a simple vector field. Let us consider a fluctuating vector $\mathbf{b}$, that is statistically steady and homogeneous. It is set to be anisotropic so that its correlation $\langle b_i b_j \rangle$ is given by

$$\langle b_i \rangle \langle b_j \rangle \neq 0, \quad \langle b_i b_j \rangle = 0,$$

in the inertial frame. When the vector is observed in a spanwise rotating system, the component is given by

$$b_i^r = Q_i b_j,$$

where

$$Q_{ij} = \begin{pmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{pmatrix}.$$  

The correlation in the rotating frame is then expressed as

$$\langle b_i^r(b_j^r) \rangle = Q_{ij} \langle b_i b_j \rangle Q_{mj}^T,$$

where $Q_{ij}^T$ is the transpose of $Q_{ij}$.

Although the correlation is constant in time in the inertial frame, it varies in time in the rotating frame as follows:

$$\langle b_i^r \rangle = \frac{1}{2} (\langle b_i \rangle + \langle b_j \rangle + \frac{1}{2} (\langle b_i \rangle - \langle b_j \rangle) \cos(2\Omega t), $$

$$\langle b_i \rangle^2 = \frac{1}{2} (\langle b_i^r \rangle + \langle b_j^r \rangle - \frac{1}{2} (\langle b_i \rangle - \langle b_j \rangle) \cos(2\Omega t), $$

$$\langle b_i^r \rangle b_j^r = -\frac{1}{2} (\langle b_i \rangle - \langle b_j \rangle) \sin(2\Omega t).$$

The material derivative $D(b_i^r(b_j^r))/Dt$ (where $U_\infty = 0$) has a nonzero value. These variations are due to the system rotation and do not mean the evolution of the correlation itself. Therefore, the material derivative does not represent the proper behavior of the correlation in the rotating frame.
On the other hand, the corotational derivative vanishes as
\[
\frac{D\langle b_i^{(r)}b_j^{(r)}\rangle}{Dt} = \frac{\partial b_i^{(r)}b_j^{(r)}}{\partial t} + \frac{\partial}{\partial t}(\frac{\partial b_i^{(r)}b_j^{(r)}}{\partial x_k} + \partial(\frac{\partial b_i^{(r)}b_j^{(r)}}{\partial x_k}))\Omega_{kj} = 0,
\]
showing that the correlation is steady in time. It can be re-written as
\[
\frac{D\langle b_i^{(r)}b_j^{(r)}\rangle}{Dt} = D_{ij}(Q_{km} \langle b_m^{(r)}b_n^{(r)}Q_{np}\rangle Q_{pj}). \quad (A9)
\]
This expression means that the correlation is transformed from the inertial frame, its material derivative is evaluated, and is then transformed back to the rotating frame. Therefore, the corotational derivative can appropriately represent the time variation of the correlation and should be used in turbulence models as a term expressing the unsteady behavior of the correlation.


