Reynolds-averaged turbulence model for magnetohydrodynamic dynamo in a rotating spherical shell

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Numerical simulation of the magnetic field in a rotating spherical shell was carried out to assess and improve the Reynolds-averaged turbulence model for magnetohydrodynamic flows. In the three-equation model the transport equations for the turbulent energy, its dissipation rate, and the turbulent helicity are solved in addition to the mean magnetic field. The turbulent electromotive force involved in the induction equation is expressed in terms of the \( \alpha \) dynamo and turbulent diffusivity terms. Since the model was improved considering the realizability condition for the turbulent electromotive force, steady state solutions were obtained even in the case of rapidly rotating system such as the Earth. Profiles of the magnetic field, the turbulent energy, and the turbulent helicity as well as their transport equations were examined to check the dynamo mechanism expressed in the model. The dependence on the system rotation and on model constants was examined to assess the model performance. © 2004 American Institute of Physics.

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I. INTRODUCTION

The dynamo effect plays an important role in magnetohydrodynamic (MHD) turbulence in plasmas and electrically conducting fluid flows. Some dynamo action is necessary for a global structure of the magnetic field to be maintained against the diffusivity. One of the typical examples is the Earth’s magnetic field.1–3 The turbulent dynamo mechanism has been studied using the mean-field theory.4,5 The turbulent electromotive force involved in the induction equation for the mean magnetic field is expressed as the product of the \( \alpha \) coefficient and the mean magnetic field; this is called the \( \alpha \) dynamo. By prescribing the value of the \( \alpha \) coefficient the sustainment of the mean magnetic field was explained. For homogeneous turbulence the \( \alpha \) coefficient can be expressed in terms of the spectra of the kinetic energy and the kinetic helicity.4,5 However, for inhomogeneous turbulence an expression for the \( \alpha \) coefficient in the physical space is necessary; a closed system of equations for the \( \alpha \) dynamo was not obtained by the mean-field theory. On the other hand, due to the rapid progress of the computer three-dimensional simulations of the magnetic field in a rotating spherical shell have been carried out.6–14 Several simulations have shown the magnetic field reversal since the first success by Glatzmaier and Roberts.8 In these simulations detailed fluctuations of the velocity and magnetic fields are calculated directly without turbulence models. However, it is still difficult to resolve small-scale fluctuations in the Earth’s core at high Rayleigh number. At present it is necessary to limit simulations to moderate Rayleigh numbers or to prescribe some artificial diffusivity.7

Another example of the dynamo effect can be found in the magnetic field in devices of the controlled fusion plasma. The reversed field pinch (RFP) is a toroidal device similar to the Tokamak.15 Since the plasma current in the RFP is very high, the velocity and magnetic fields are expected to be in a turbulent state. Using a relaxation theory Taylor16 clarified the global magnetic structure in the sustainment phase. Gillette and Watkins17 showed that the Taylor state can be explained in terms of the \( \alpha \) dynamo effect. Three-dimensional simulations for the RFP were also carried out.18

The difficulty in computing high Rayleigh or Reynolds number flows also exists in simulating non-MHD turbulence; turbulence modeling has been studied in the engineering field. There are two types of turbulence modeling according to averaging. In the large eddy simulation (LES) filtering is applied to divide the velocity into the grid-scale (GS) and the subgrid-scale (SGS) components.19 The GS velocity is solved explicitly whereas the effect of the SGS velocity on the GS velocity needs to be modeled. Although computational cost is high, the SGS model is expected to be simple and universal. On the other hand, ensemble averaging is applied in the Reynolds-averaged model.20 Computational cost is low because one- or two-dimensional simulations can be carried out for a turbulent field homogeneous in some directions. Instead, model expressions are rather complicated. Recently the LES has begun to be applied to practical engineering flows and is expected to be a main tool for the computational fluid dynamics in near future. However, it requires very high computing cost for solving wall-bounded turbulent flows because very small eddies near the wall need to be resolved with a fine grid. The Reynolds-averaged model is still used in many areas such as the weather forecast21 and the design of aircraft wings.22 Moreover, hybrid Reynolds-averaged/LES approach was proposed and studied to develop better wall modeling for the LES; the Reynolds-averaged model is solved near the wall whereas the LES is carried out away from the wall.22,23 Therefore, it is still necessary and useful to develop better Reynolds-averaged models.
In non-MHD turbulence the main problem of turbulence modeling is to model the turbulent viscosity for the mean velocity. The counterpart in MHD turbulence is the $\beta$ effect or the turbulent diffusivity for the magnetic field. In addition, the dynamo effect needs to be modeled in some form. The Reynolds-averaged model for MHD turbulence has been investigated theoretically. The $K$-$\varepsilon$ model is a widely used Reynolds-averaged model for non-MHD turbulence where $K$ is the turbulent energy and $\varepsilon$ is its dissipation rate. Using a statistical theory called the two-scale direct interaction approximation Yoshizawa$^{26}$ extended the $K$-$\varepsilon$ model to MHD flows. The ultimate goal of this theoretical work is to develop a universal MHD turbulence model. The proposed model was first applied to controlled fusion plasma and was recently used for analysis of astrophysical/geophysical turbulence. For example, to examine the $\alpha$ dynamo in the RFP Yoshizawa and Hamba$^{27}$ proposed a three-equation model treating the transport equations for $K$, $\varepsilon$, and the turbulent helicity. Yoshizawa$^{28}$ paid attention to the cross helicity, the correlation of the velocity and magnetic field fluctuations, and showed that it is important for the MHD turbulence accompanied with the mean flow. Using the four-equation model involving the cross helicity Yoshizawa $et$ $al.$$^{29}$ analytically explained the imbalance between the kinetic and magnetic energies in the Earth’s core and Yoshizawa $et$ $al.$$^{30}$ theoretically examined the solar polarity reversal.

Although these works show the potential of the proposed model, most of them are qualitative analysis. The theory is based on several assumptions and model equations involve several model constants as usual Reynolds-averaged models for non-MHD turbulence. In order to make the model reliable, it is necessary to assess the model comparing its simulation results with experiment and observation as well as with other three-dimensional simulations. Hamba$^{31}$ carried out a three-dimensional simulation of MHD turbulence in a cube to examine the $\alpha$ and cross-helicity dynamo effects. Using the three-equation model Hamba$^{32}$ carried out a one-dimensional simulation of the RFP; it was shown that the result of the Reynolds-averaged model agrees fairly well with the result of the three-dimensional LES of the RFP. The MHD turbulence in the RFP is sustained by the external electric field whereas that in the Earth’s core is caused by the buoyancy force and the system rotation. Since the mechanism is different it is very interesting and important to apply the model to turbulent convection in a rotating spherical shell to check whether the model can reproduce the large-scale magnetic field. In this case the Reynolds-averaged model is similar to 2.5-dimensional simulations of the Earth’s magnetic field$^{33}$ in the sense that time evolution of a few quantities in addition to the mean field is solved without high computing cost. For example, Sarson $et$ $al.$$^{34}$ calculated a single nonaxisymmetric mode whereas the Reynolds-averaged model treats a few statistical quantities representing all nonaxisymmetric modes.

Using the same model for the RFP the author previously tried to calculate the Earth’s magnetic field; it was found that the calculation overflow and a solution was not obtained. This failure is due to very rapid rotation of the Earth; its angular velocity is about $10^5$ when normalized by the radius and the velocity scale of the core. The turbulence model needs to be improved so that it can be applied to flows in rapidly rotating systems. In non-MHD turbulence the performance of the turbulence model for high mean shear has been improved by considering the realizability condition for the Reynolds stress; this condition requires the absolute value of the Reynolds shear stress be less than the turbulent energy. In the present work we pay attention to the realizability condition for the turbulent electromotive force in order to improve the model for the $\alpha$ dynamo and to obtain a solution to the model expression for rapidly rotating systems.

We should note that the objective of the present work is not to develop a turbulence model that is very accurate but is applicable only to the Earth’s dynamo. In order to attack the Earth’s dynamo problem it is better to carry out three-dimensional simulations. Instead we are interested in the dynamo effect common in various MHD turbulence such as the RFP, the Earth, the Sun, and other astrophysical/geophysical and engineering flows. We try to develop a general model to explain the dynamo effect or the sustainment of global magnetic field due to the motion of electrically conducting fluid. Since the overall magnetic field of the RFP can be expressed with our model, we then apply the model to the geodynamo problem as another benchmark for the general turbulence model. The method of the present simulation consists of two parts. The first part is model equations. We believe that we can construct a general turbulence model and that its model equations including values of model constants can be applied to various MHD turbulence. The second part is a flow geometry, boundary conditions, and physical parameters including the magnetic diffusivity; they are specific to the Earth. Those properties are set close to those for the Earth in order to compare the result of this simulation with observation and three-dimensional simulations. We will assess whether the sustainment of the Earth’s large-scale magnetic field can be reproduced as a first approximation.

Let us mention the relationship between the present Reynolds-averaged model and three-dimensional simulations developed recently. Three-dimensional simulations without turbulence models are more reliable because they have less assumptions compared to the Reynolds-averaged model. However, due to the grid resolution requirement three-dimensional simulations are limited to Rayleigh numbers much less than that for the Earth. It is not clear to what extent results of simulations with low Rayleigh numbers can reproduce the Earth’s magnetic field. On the other hand, the Reynolds-averaged model is expected to simulate high Rayleigh number flows successfully if the effect of small-scale turbulence is appropriately modeled. At present, the Reynolds-averaged model is not yet very reliable; this is why we try to apply the model to various MHD turbulence. We believe that in future the Reynolds-averaged model can be more reliable and can contribute to astrophysical/geophysical and engineering research by compensating three-dimensional simulations.

The paper is organized as follows. In the following section we explain the three-equation model for MHD turbulence. Some coefficients are introduced considering the realizability condition for the turbulent electromotive force. In
Sec. III we describe the procedure of the simulation of the magnetic field in a rotating spherical shell. In Sec. IV we show profiles of the manganic field, the turbulent energy, and the turbulent helicity to assess the model performance. The dependence on the system rotation and on model constants is also examined. Concluding remarks are given in Sec. V.

II. MODEL EQUATIONS

In this paper we adopt Alfvén velocity units and replace the magnetic field \( b/\sqrt{\rho \mu_0} \rightarrow b \), the electric current density \( j/\sqrt{\rho \mu_0} \rightarrow j \), and the electric field \( e/\sqrt{\rho \mu_0} \rightarrow e \) where \( \rho \) is the fluid density and \( \mu_0 \) is the magnetic permeability. We use ensemble averaging \( \langle \cdot \rangle \) to divide a quantity into the mean and fluctuating parts as

\[
f = F + f', \quad F = \langle f \rangle,\]

where \( f = (u, b, e, j, \omega) \). The induction equation for the mean magnetic field \( B \) is written as

\[
\frac{\partial B}{\partial t} = - \nabla \times E.\]

Here, the magnetic field also satisfies \( \nabla \cdot B = 0 \) and the mean electric field \( E \) is given by

\[
E = -\mathbf{U} \times B - E_M + \eta \mathbf{J},\]

where \( \mathbf{U} \) is the mean velocity, \( E_M(=\langle \mathbf{u}' \times \mathbf{b}' \rangle) \) is the turbulent electromotive force, \( \eta \) is the magnetic diffusivity, and \( \mathbf{J}(=\nabla \times \mathbf{B}) \) is the mean current density. In order to calculate the time evolution of the mean magnetic field we need to evaluate \( \mathbf{U} \) and \( E_M \), the mean velocity is obtained by solving the averaged Navier-Stokes equation whereas some modeling is needed for the turbulent electromotive force. In this work we assume that the mean velocity is zero. The reason for this assumption will be mentioned later when the numerical procedure is described.

A model for the turbulent electromotive force can be given by

\[
E_M = \alpha \mathbf{B} - \beta \mathbf{J},\]

where the first term on the right-hand side represents the \( \alpha \) dynamo whereas \( \beta \) in the second term is the turbulent magnetic diffusivity. For homogeneous turbulence an expression for the coefficient \( \alpha \) was derived using the energy and helicity spectra in the wavenumber space in the mean-field theory. However, for inhomogeneous turbulence the coefficient \( \alpha \) needs to be expressed in terms of quantities in the physical space. In this work we adopt the three-equation turbulence model, it was derived by extending the \( K-\varepsilon \) model to MHD turbulence. The fluid density \( \rho \) is assumed to be uniform except for the buoyancy force. As basic variables we use the turbulent energy \( K \), its dissipation rate \( \varepsilon \), and the turbulent helicity \( H \) defined as

\[
K = \frac{1}{2} \langle u^2 + b'^2 \rangle,\]

\[
\varepsilon = \nu \langle s_{ij}^2 \rangle,\]

\[
H = \langle -u' \cdot \omega' + b' \cdot j' \rangle,\]

respectively, where

\[
s_{ij}' = \frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}, \quad \omega' = \nabla \times \mathbf{u}', \quad j' = \nabla \times \mathbf{b}'.\]

Using the three variables the coefficients in Eq. (4) can be modeled as

\[
\alpha = C_\alpha f_a \frac{K H}{\varepsilon}, \quad \beta = C_\beta f_\beta \frac{K^2}{\varepsilon},\]

where \( f_a \) and \( f_\beta \) are correction coefficients given by

\[
f_a = \left[ 1 + \left( C_\alpha \frac{H}{\varepsilon} \right)^2 \right]^{-1/2},\]

\[
f_\beta = \left[ 1 + \left( C_\beta \frac{J}{\varepsilon} \right)^2 \right]^{-1/2},\]

and \( C_\alpha, C_\beta, C_{\beta 1}, \) and \( C_{\beta 2} \) are nondimensional model constants. In the previous model correction coefficients were not used \( (f_a = f_\beta = 1) \). Since \( \alpha \) and \( H \) are pseudoscalars \( \alpha \) is proportional to \( H \). Expressions in Eq. (7) except for \( f_a \) and \( f_\beta \) can be obtained by dimensional analysis. In the present model the correction coefficients are introduced in order not to overestimate \( E_M \).

To close the system of equations we treat the transport equations for \( K, \varepsilon, \) and \( H \). The equation for \( K \) is given by

\[
\frac{\partial K}{\partial t} = P_{KM} + P_{KB} - \varepsilon + \nabla \cdot \mathbf{T}_K,\]

where

\[
P_{KM} = -E_M \cdot J, \quad P_{KB} = -\alpha_1 \mathbf{g} \cdot \langle \theta' \mathbf{u}' \rangle, \quad \mathbf{T}_K = \frac{\nu_s}{\sigma_K} \nabla K, \quad \nu_s = C_s \frac{K^2}{\varepsilon}.\]

Here, \( \alpha_1 \) is the thermal expansion coefficient, \( \mathbf{g} \) is the gravity acceleration, \( \langle \theta' \mathbf{u}' \rangle \) is the turbulent heat flux, and \( \sigma_K \) and \( C_s \) are model constants. The right-hand side of Eq. (9) consists of four terms. The magnetic production term \( P_{KM} \) represents the interaction between the turbulent energy \( K \) and the energy of the mean magnetic field, \( \mathbf{B}^2/2 \). If \( P_{KM} \) is positive the energy is cascaded from \( \mathbf{B}^2/2 \) to \( K \); if it is negative the energy is transferred in the opposite direction as a dynamo action. The direction is determined by the balance between the dynamo part \( -\alpha \mathbf{B} \cdot \mathbf{J} \) and the turbulent diffusivity part \( \beta \mathbf{J}^2 \). The buoyant production term \( P_{KB} \) is positive for unstably stratified turbulent field. In this work the value of \( P_{KB} \) is prescribed instead of solving the mean temperature equation. The third term \( \varepsilon \) on the right-hand side denotes the energy dissipation. It involves the dissipation of the turbulent kinetic energy \( \langle u'^2 \rangle/2 \) due to viscosity and the loss of the turbulent magnetic energy \( \langle b'^2 \rangle/2 \) due to Joule heating. The fourth
term $\nabla \cdot T_K$ is the turbulent diffusion. It represents the energy transfer from a point to another point in the physical space. The diffusivity coefficient for this transfer is assumed to be proportional to the turbulent viscosity $\nu_T$.

The transport equations for $e$ and $H$ are given by

$$\frac{\partial e}{\partial t} = C_{e_1} \left( P_{KM} + P_{KB} \right) - C_{e_2} \frac{e^2}{K} - \nabla \cdot \left( \frac{\nu_T}{\sigma_e} \nabla e \right),$$

$$\frac{\partial H}{\partial t} = -C_{H_1} \frac{e^2}{K^2} E_M \cdot B - C_{H_2} \frac{eH}{K} + \nabla \cdot \left( -K \Omega_0 + \frac{\nu_T}{\sigma_H} \nabla H \right),$$

respectively, where $C_{e_1}$, $C_{e_2}$, $\sigma_e$, $C_{H_1}$, $C_{H_2}$, and $\sigma_H$ are model constants and $\Omega_0$ is the angular velocity vector of the system rotation. Each equation consists of the production part, the dissipation part, and the diffusion part. The diffusion part in the $H$ equation involves not only the gradient-diffusion term $\nabla \cdot (\nu_T/\sigma_H \nabla H)$ but also the mean vorticity term $\nabla \cdot (-K \Omega_0)$. We note that in the case of nonzero mean velocity, $\Omega_0$ should be replaced by $\Omega/2 + \Omega_0$ where $\Omega(\nabla \times U)$ is the mean vorticity. Since $\Omega_0$ is constant in space the term involved in Eq. (12) is rewritten as $-\Omega_0 \nabla K$. If $\partial K/\partial r < 0$ is satisfied in a rotating spherical shell the term is positive (negative) and produces positive (negative) turbulent helicity $H$ in the northern (southern) hemisphere.

The three-equation model with $P_{KB}=0$ and $f_a=f_\beta=1$ was successfully applied to the simulation of the magnetic field in the RFP of controlled fusion plasma. At the same time we also tried to apply this model to the Earth’s magnetic field. It was found that the calculation overflows; the magnetic field grew too quickly and no steady state solution was obtained. This is because the rotation of the Earth is too rapid compared to the turbulent time scale; the term $\nabla \cdot (-K \Omega_0)$ produces a large value of the turbulent helicity and overestimates the $\alpha$ coefficient. In this work we introduced the correction coefficient $f_a$ to overcome this difficulty. In obtaining expressions for $f_a$ and $f_\beta$ we considered the realizability condition for the turbulent electromotive force $E_M$.

In the non-MHD turbulence the realizability conditions for the Reynolds stress $(u'_i u'_j)$ are examined to improve the eddy viscosity model. The realizability conditions are given by

$$\langle u'^2 \rangle \geq 0, \quad \langle u'_i u'_j \rangle^2 \leq \langle u'^2 \rangle (u'^2).$$

Here summation convention is not applied for $i$ and $j$. The second condition is Schwarz’ inequality for the shear stress. Similarly, the realizability condition for the turbulent electromotive force can be given by

$$\langle (u' \times b')^2 \rangle \leq \langle u'^2 \rangle \langle b'^2 \rangle \leq \langle (u'^2 + b'^2)/2 \rangle^2.$$  

This leads to the relation between $E_M$ and $K$ as

$$\frac{E_M}{K} = \frac{\langle (u' \times b')^2 \rangle}{\langle (u'^2 + b'^2)/2 \rangle} \leq 1.$$  

If $f_a=1$ this condition is not necessarily satisfied for a large value of $H$. We set the expression for $f_a$ in Eq. (8) so that the $\alpha$ term cannot violate condition (15). Actually, we can show that

$$\frac{|\alpha B|}{K} = \sqrt{\frac{C_{a1} H}{\epsilon}} \leq \sqrt{1 + \frac{C_{a2} H^2}{\epsilon}} \approx 1$$

if $|C_{a1}| \leq |C_{a2}|$. The expression for $f_\beta$ was obtained in the same manner. Coefficients similar to $f_\beta$ are also used for modeling the turbulent viscosity in non-MHD turbulence. In this case the mean rate $S_j = (\partial U_j/\partial x_i + \partial U_i/\partial x_j)$ or the mean vorticity $\Omega$ is used instead of $J$. This modification for the turbulent viscosity was introduced by considering the realizability condition for the Reynolds stress model.

We should note that expressions for $f_a$ and $f_\beta$ in Eq. (8) are not uniquely determined; they are just candidates for models satisfying Eq. (15). For example, the following expression for $f_a$ is also valid:

$$f_a = \left[ 1 + \left( C_{a2} \frac{H}{\epsilon} \right)^2 \right]^{-n},$$

where $n > 1/2$. In this case the value of $\alpha$ tends to zero as $H$ increases to infinity. In the present work we assume that even if $H$ is very large the $\alpha$ term has a finite value and balances to the $\beta$ term. Therefore we adopted $n=1/2$ in Eq. (8).

For the $\alpha$ effect a similar modification called the $\alpha$ quenching was derived by Gruzinov and Diamond. The $\alpha$ coefficient is modified as

$$\alpha = a_0 (1 + R_m B_0^2 u'^2)^{-1},$$

where $a_0$ is the value due to the kinematic helicity, $R_m$ is the magnetic Reynolds number, $B_0$ is the large-scale magnetic field, and $v$ is the characteristic turbulent velocity. The $\alpha$ effect is suppressed when the mean magnetic field is large; as $B_0$ increases to infinity the value of $\alpha B$ decays to zero. On the other hand, in our model $\alpha B$ tends to a finite value of the order of $K$. Moreover, the correction coefficients $f_a$ and $f_\beta$ are effective not only for a large magnetic field but also for a large value of $H$; this property is necessary for steady state solutions to be obtained in the case of rapidly rotating system.

Here, we mention the effect of the system rotation on $K$ and $e$. As shown in Eqs. (9) and (11) the angular velocity $\Omega_0$ is not explicitly involved in the $K$ and $e$ equations unlike the $H$ equation. The rotation effect is incorporated through $P_{KM} = -E_M \cdot J$ because the $\alpha$ term in $E_M$ depends on $H$. Therefore, for non-MHD turbulence the present $K$ and $e$ equations are independent of $\Omega_0$ because $P_{KM}$ vanishes; this model cannot explain the rotation effect such as the anisotropy of turbulent field due to convection cells parallel to the rotation axis. Such a convecting structure may partly be reproduced if the mean velocity is treated. Another candidate for modeling the rotation effect is the introduction of the anisotropic diffusivity tensor. Braginsky and Meytis showed that the influence of the Earth’s rotation and of the magnetic field makes the small scale turbulence highly anisotropic. In the present model the turbulent diffusivity such as $\nu_T/\sigma_k$ is a scalar representing isotropic diffusivity. In fu-
tute the diffusivity tensor should be modeled using a tensor such as \( \Omega_{ij} \Omega_{ij} \) and taking the insight of the local turbulence into account.

III. NUMERICAL SIMULATION

In this section we explain the procedure of the simulation of the magnetic field in a rotating spherical shell. In the direct numerical simulation and in the LES three-dimensional calculations are carried out; physical quantities depend on all three coordinates in space. In this work, to assess the Reynolds-averaged model we apply ensemble averaging to the magnetic field equation. In this averaging mean quantities are independent of coordinates in statistically homogeneous directions; calculations can be reduced to one or two dimensions in general. For example, for a fully developed turbulent flow in a straight pipe the mean velocity is independent of the streamwise and azimuthal coordinates and depends only on the radial coordinate; its calculation is one-dimensional. If the radius of the pipe changes in the streamwise direction, the mean velocity also depends on the streamwise coordinate; the simulation becomes two-dimensional. Similarly, for a rotating spherical shell adopted in this work mean quantities are independent of \( \phi \) and depend on \( r \) and \( \theta \) in spherical coordinates. This dependence implies that we solve axisymmetric modes as the mean field and treat nonaxisymmetric modes as the fluctuating field.

Figure 1 shows the two-dimensional computational domain of a rotating spherical shell corresponding to the Earth’s outer core. The regions outside the shell are assumed to be insulators for simplicity. It is more realistic to assume that the electrical conductivity of the inner core is similar to that of the outer core; the effect of its conductivity on the magnetic field was studied in three-dimensional simulations. Nevertheless, we believe that fundamental processes of MHD dynamo can be captured and examined with this simplified boundary conditions. Within the framework of the present model and simplification, we set values of parameters considering the properties of the Earth. For example, the ratio of radii of the inner and outer boundaries is set to \( R_{in}/R_{out} = 0.35 \).

As already mentioned we assume that the mean velocity is zero. This does not necessarily imply that the mean velocity is negligible in a rotating spherical shell. There are two reasons for this assumption. First, the model equations for self-consistent treatment are complicated. In order to solve the mean velocity equation a model for the Reynolds stress \( \langle u' u' - b' b' \rangle \) is necessary. A model for the turbulent heat flux \( \langle \theta' u' \rangle \) is also required to solve the mean temperature equation. The present three-equation model involves 12 model constants. When the number of model expressions and constants increases further, it is difficult to assess the model performance at one time. Therefore, as a first step toward a self-consistent treatment with the Reynolds-averaged turbulence model we solve only the equations for \( B \) as well as \( \alpha \), \( \epsilon \), and \( H \). Second, nonaxisymmetric velocity modes are more important than axisymmetric modes as a dynamo action. Cowling’s antidynamo theorem says that axisymmetric magnetic field cannot be maintained by axisymmetric fluid motion; nonaxisymmetric modes are needed to explain the dynamo mechanism. In their three-dimensional simulations Olson et al. showed that the axisymmetric azimuthal flow is weak in the case of high Rayleigh number. In the present averaging axisymmetric and nonaxisymmetric modes are considered the mean and fluctuating fields, respectively. We treat the fluctuating fluid motion through \( K \) and \( H \); we expect that those variables represent the effect of nonaxisymmetric modes.

This assumption for the mean velocity is closely related to the type of the dynamo mechanism. The differential rotation contributing to the \( \omega \) dynamo can be expressed in terms of the gradient of the mean toroidal velocity \( \omega \) in the poloidal directions. On the other hand, small-scale helical motion related to the \( \alpha \) dynamo is modeled in terms of the turbulent helicity \( H \) in this simulation. Therefore, the present simulation cannot reproduce the \( \alpha \)-dynamo but the \( \alpha^2 \) dynamo. Since the work of Parker the mean-field theory mainly studied the \( \omega \)-dynamo. We do not insist that the \( \alpha^2 \) dynamo is more appropriate as the Earth’s dynamo; in future work we need to solve the mean velocity equation to reproduce the \( \alpha \omega \) dynamo and to compare the two dynamo effects. Nevertheless, we believe that the present simulation is not unphysical because a three-dimensional simulation suggested that \( \alpha^2 \) dynamo is also possible.

The variables we treat are the three components of the mean magnetic field, \( B_r \) and \( B_\theta \) as well as the turbulent energy \( K \), its dissipation rate \( \epsilon \), and the turbulent helicity \( H \). From Eqs. (2) to (4) and the assumption of \( U = 0 \), we can write the induction equation for \( B \) as
\[
\frac{\partial B}{\partial t} = \nabla \times [aB - (\beta + \eta)J].
\]  

(19)

Coefficients \(\alpha\) and \(\beta\) are expressed in terms of \(K\), \(\epsilon\), and \(H\) as shown in Eq. (7). The transport equations for \(K\), \(\epsilon\), and \(H\) are given by Eqs. (9), (11), and (12), respectively.

Physical quantities are nondimensionalized by the typical velocity \(U_0=5\times10^{-4}\) m/s, the radius of the outer boundary \(R_{\text{out}}=3.48\times10^5\) m, and the magnetic permeability \(\mu_0=4\pi\times10^{-7}\) Vs A\(^{-1}\) m\(^{-1}\). The value of \(U_0\) is the characteristic velocity based on the westward drift of the field patterns. Kono and Roberts\(^3\) discussed that the heat flux at the core-mantle boundary is different for the inner and outer boundaries. The reason is explained as follows. As a first approximation it is expected that the buoyant production term and the dissipation term balance each other in the \(K\) equation. In their three-dimensional simulations Olson et al.\(^{11}\) found that the two terms nearly balance locally in some cases. This balance implies that \(\epsilon=\epsilon_K\) is a reasonable condition. On the other hand, the free-slip condition corresponds to \(\epsilon_r=\epsilon_0=0\). In a preliminary calculation of \(K\) and \(\epsilon\) with \(B\)=0 we tried the following four cases: \(\epsilon=\epsilon_K\) or \(\epsilon_r=\epsilon_0=0\) at \(r=R_{\text{in}}\) and \(r=R_{\text{out}}\). A steady solution was obtained only for the condition given in Eq. (24). In the other three cases the value of \(K\) continued to grow; a steady state was not obtained. Therefore, we adopt the boundary conditions for \(\epsilon\) in Eq. (24) in the present simulation.

The initial condition for the magnetic field is given by

\[
B_0 = B_\phi = 0, \quad B_\theta = -\exp[-(r-r_1)^2/r_2^2]\sin 2\theta.
\]  

(25)

where \(r_1=(R_{\text{in}}+R_{\text{out}})/2\) and \(r_2=(R_{\text{out}}-R_{\text{in}})/4\). A seed magnetic field is imposed for \(B_\phi\); its intensity does not affect the steady state solution but the symmetry with respect to the equator is important. In this simulation \(B_\phi\) is antisymmetric; it is negative in the northern hemisphere and positive in the southern hemisphere. As will be shown later such a profile causes a negative magnetic dipole moment corresponding to the present Earth. The initial conditions for the other quantities are given by

\[
K = 900, \quad \epsilon = \epsilon_K, \quad H = 0.
\]  

(26)

The condition for \(\epsilon\) is set considering the balance between the production and dissipation terms in the \(K\) equation. The value of \(K\) is roughly estimated as \((\epsilon R_{\text{out}})^{2/3}\) at \(r=R_{\text{in}}\).

We solve the equation for \(B\) as well as the three-equation model. The uniform grid is used; the grid number is set to \(N_r\times N_\theta=52\times48\). The second-order finite difference scheme is adopted in space. For time integration the Adams-Bashforth scheme is used; the interval of time step is set to \(\Delta t=3\times10^{-6}\).

IV. RESULTS AND DISCUSSION

In this section we show results of the simulations and examine the dynamo mechanism expressed in the present model. First, we mention values of physical parameters important for the geodynamo problem. In this simulation the kinematic viscosity \(\nu\) and the thermal diffusivity \(\kappa\) are not treated explicitly. Like the magnetic diffusivity \(\eta\), the parameters \(\nu\) and \(\kappa\) are much less than the turbulent viscosity \(\nu_T\).
and the turbulent thermal diffusivity $\kappa_T$, respectively. Therefore, the Rayleigh number $Ra = \frac{\alpha \beta_0 \rho R_{\text{out}}^2}{\nu \kappa}$ where $\beta_0$ is the temperature gradient at the upper boundary) and the Taylor number $Ta = \frac{2 \Omega (R^2_{\text{out}}/\nu)^2}{\nu \kappa}$ are effectively infinity; the Prandtl number $Pr = (\nu/\kappa)$ and the magnetic Prandtl number $Pm = (\nu/\eta)$ are of order unity. To be more accurate let us substitute the actual values of $\nu$ and $\kappa$ to the definitions of $Ra$ and $Ta$. We assumed that the value of the heat flux at $r = R_{\text{out}}$ is $2.2 \times 10^{12}$ W; this value corresponds to the temperature gradient of $\beta_0 = 3.4 \times 10^{-4}$ K m$^{-1}$. Substituting the values of $\beta_0$ and other quantities we have $Ra = 2 \times 10^{30}$. Similarly, the highest rotation rate $\Omega = 5 \times 10^5$ adopted later gives the Taylor number $Ta = 3 \times 10^{30}$. The values of $Ra$ and $Ta$ are of the same order as those for the Earth. On the other hand, Kono and Roberts$^3$ reported that typical three-dimensional simulations with the hyperdiffusivity were carried out at $Ra = 10^7$ and those with no hyperdiffusivity were at $Ra = 10^3$–$10^7$. The above estimate indicates that the Reynolds-averaged model can simulate turbulence at high Rayleigh numbers that cannot be applied in three-dimensional direct numerical simulations. Of course, for the Reynolds-averaged model to be reliable the small-scale turbulence needs to be modeled appropriately.

To solve the model equations we need to set values of the model constants; we mainly use the following two sets of constants:

**case 1:**

\[
\begin{align*}
C_{a1} &= 0.09, \quad C_{a2} = 0.18, \quad C_{\beta_1} = 0.09, \\
C_{\beta_2} &= 0.09, \quad C_T = 0.09, \quad \sigma_K = 1, \\
C_{e_1} &= 1.4, \quad C_{e_2} = 1.9, \quad \sigma_\varepsilon = 1.3, \\
C_{H_1} &= 1, \quad C_{H_2} = 1, \quad \sigma_H = 1, \\
\end{align*}
\]

(27)

**case 2:**

\[
\begin{align*}
C_{a1} &= 0.14, \quad C_{a2} = 0.28, \quad C_{\beta_1} = 0.14, \\
C_{\beta_2} &= 0.14, \quad C_T = 0.09, \quad \sigma_K = 1, \\
C_{e_1} &= 1.8, \quad C_{e_2} = 1.9, \quad \sigma_\varepsilon = 1.3, \\
C_{H_1} &= 5, \quad C_{H_2} = 1, \quad \sigma_H = 0.6. \\
\end{align*}
\]

(28)

Case 1 is used as the standard set in this simulation. Among 12 constants $C_T$, $\sigma_\varepsilon$, $C_{e_1}$, $C_{e_2}$, and $\sigma_\varepsilon$ are also involved in the $K$-$\varepsilon$ model for non-MHD turbulence. We use the same values of the five constants as those in the standard $K$-$\varepsilon$ model.\textsuperscript{20} The $\alpha$ dynamo term and the turbulent diffusivity term are expected to balance each other in the turbulent electromotive force and the turbulent diffusivity $\beta$ plays a similar role to the turbulent viscosity; constants $C_{a1}$ and $C_{\beta_1}$ for the $a$ and $\beta$ terms are then set equal to $C_T$ for the turbulent viscosity. The behavior of the $H$ equation was not fully understood; $C_{H_1}$, $C_{H_2}$, and $\sigma_H$ are set to be unity as a first approximation. On the other hand, the values of constants in case 2 except for $C_{a2}$ and $C_{\beta_2}$ were the same as those adopted in the simulation of the RFP.\textsuperscript{32} Some of the constants in case 2 have different values from those in case 1. Profiles of the magnetic field in the RFP agree fairly well with results of three-dimensional simulation.\textsuperscript{32} If Earth’s magnetic field is also reproduced, we can expect that the three-equation model is a model not only for the RFP but also for general MHD turbulence. In both cases $C_{\beta_2}$ is set equal to $C_{\beta_1}$ considering a condition similar to Eq. (16). Constant $C_{a2}$ is set to be $C_{a1}/2$; this is because a steady state was not obtained for $C_{a2} = C_{a1}$.

We should note that we do not seek values of model constants applicable only to the Earth’s dynamo; we try to construct a general turbulence model including values of model constants. For example, in the non-MHD turbulence the value of $C_T = 0.09$ for the eddy viscosity is widely used in various types of turbulence including wall-bounded flows and free shear layers. The values of $C_T$, $\sigma_K$, $C_{e_1}$, $C_{e_2}$, and $\sigma_\varepsilon$ adopted in case 1 are known as a standard set in the non-MHD turbulence although some variation exists such as $C_{e_1} = 1.4$ to 1.5. We expect that we can find a standard set for the MHD turbulence. Unfortunately, in the previous simulation of the RFP we did not adopt the values in case 1 but those in case 2. In the present work we need to examine how different the result in case 2 is compared to that in case 1. In near future we should also carry out a simulation of the RFP using the values in case 1 to see whether the values are also valid for the RFP.

Since the present model involves many adjustable constants one may consider it unreliable. However, to have many constants is not unusual as the Reynolds-averaged model. For example, the Mellor-Yamada level 2 model is widely used for the simulation of the atmospheric boundary layer and the weather forecast.\textsuperscript{21} The model involves five constants and their higher-level model has more. The Spalart-Allmaras model used for the simulation of flows around aircraft wings involves seven constants.\textsuperscript{22} The standard $K$-$\varepsilon$ model adopted here is widely used in mechanical engineering; it involves five constants and its improved version called the nonlinear $K$-$\varepsilon$ model needs more constants. Of course, in order for such a model to be reliable the values of constants need to be optimized by comparing its results with experiment and three-dimensional simulations. In the present model the number of constants is rather large because we also treat the magnetic field. We need to apply the model to various MHD turbulence in order to make it reliable. This is why we simulate the Earth’s magnetic field as a typical example of the turbulent dynamo effect; the dipole magnetic field should be reproduced as a first approximation. Much more work needs to be done to improve the present model. Nevertheless, we believe the present work is meaningful as a first step to develop a general MHD turbulence model.

First, we examine the dependence on the angular velocity of the system rotation, $\Omega_0(=\Omega_0)$. The value of $\Omega_0 = 5 \times 10^5$ is nearly equal to that for the Earth. Figure 2 shows the time evolution of the volume averaged energy of the mean magnetic field given by
FIG. 2. Time evolution of the volume averaged energy of mean-magnetic field for four runs in case 1 and one run in case 2.

\[
\left( \frac{1}{2} \mathbf{B}^2 \right)_v = \frac{3}{2} \int_{R_{\text{in}}}^{R_{\text{out}}} d r \int_0^\pi d \theta \sin \theta \frac{1}{2} \mathbf{B}^2.
\]

Results of five runs are plotted: \( \Omega_0 = 5 \times 10^5, 1 \times 10^5, 3 \times 10^4, \) and \( 2 \times 10^4 \) for case 1 and \( \Omega_0 = 5 \times 10^5 \) for case 2. Unlike previous simulations with \( f_a = f_\beta = 1, \) a steady state is achieved even for the largest value of \( \Omega_0 = 5 \times 10^5. \) The magnetic energy in the steady state increases with \( \Omega_0. \) This tendency is because a large value of \( \Omega_0 \) produces much turbulent helicity and the resulting \( \alpha \) effect is large. The difference between the results for \( \Omega_0 = 5 \times 10^5 \) and \( \Omega_0 = 1 \times 10^5 \) is smaller than that between \( \Omega_0 = 1 \times 10^5 \) and \( \Omega_0 = 3 \times 10^4. \) This means that due to the correction coefficients \( f_a \) and \( f_\beta, \) the solution becomes less sensitive to \( \Omega_0 \) as \( \Omega_0 \) increases. In the case of \( \Omega_0 = 1 \times 10^4 \) the magnetic energy decays to zero (not shown here). This decay suggests that the critical value of \( \Omega_0 \) in case 1 is a little less than \( 2 \times 10^4. \) Since the nondimensional value of \( \Omega_0 \) is based on \( U_0 \) and \( L, \) it depends on the definition of \( U_0 \) and it is not clear whether \( L \) is the appropriate length scale. Another definition can be the rotation rate based on the turbulence time scale, \( \Omega = \Omega_0 K / e. \) If the volume averaged value is used for \( K \) and \( e \) then the values of \( \Omega_0 = 5 \times 10^5, 1 \times 10^5, 3 \times 10^4, \) and \( 2 \times 10^4 \) for case 1 correspond to \( \Omega = 3.3 \times 10^4, 6.7 \times 10^3, 2.0 \times 10^3, \) and \( 1.3 \times 10^3, \) respectively. For case 2 the value of \( \Omega_0 = 5 \times 10^5 \) gives \( \Omega = 1.9 \times 10^4. \)

Next, we examine profiles of several quantities for \( \Omega_0 = 5 \times 10^5 \) in case 1 in detail. Figure 3 shows profiles of the magnetic field at \( t = 1.5 \) for \( \Omega_0 = 5 \times 10^5 \) in case 1. In Fig. 3(a) the toroidal field \( B_\phi \) is negative in the northern hemisphere and positive in the southern hemisphere reflecting the initial condition given by Eq. (25). Figure 3(b) shows the lines of force of the poloidal field \( (B_r, B_\theta); \) its direction is counter-clockwise. The lines of force are the contour plots of \( r \sin \theta A_\phi \) where \( A \) is the vector potential \( (B = \nabla \times A). \) The magnetic field is similar to the longitudinally averaged profiles obtained from the three-dimensional simulation by Kuang and Bloxham. Each component has a fairly simple structure; the toroidal field is seen throughout the outer core and the poloidal field is a predominantly dipolar one. On the other hand, the simulation by Glatzmaier and Roberts shows more complicated structure. The toroidal field is concentrated inside the tangent cylinder, the cylinder drawn parallel to the rotation axis and tangent to the inner boundary, whereas the poloidal field shows strong closed loops near the inner boundary in addition to the dipolar field. As was discussed by Kuang and Bloxham, the difference in structure can be due to the difference in the boundary condition. In their and our simulations the stress-free conditions are used whereas Glatzmaier and Roberts used the no-slip conditions.

To assess the present result quantitatively we examine the dipole moment defined as

\[
\mathbf{m} = \frac{1}{2} \int (\mathbf{r} \times \mathbf{J}) dV.
\]

The absolute value of the axial component \( m_z \) is 52 in the present normalization and \( 1 \times 10^{23} \) Am\(^2\) in dimensional unit. The observed value of the Earth\(^2\) is about \( 8 \times 10^{22} \) Am\(^2\). Considering the assumption of zero mean velocity and the ambiguity of model constants we can see that the agreement between the present result and the observed value is fairly good.

Figure 4 shows the contour plots of the turbulent energy \( K \) and the turbulent helicity \( H \) at \( t = 1.5 \) for \( \Omega_0 = 5 \times 10^5 \) in case 1. In Fig. 4(a) the peak of \( K \) is located at a point on the equator near the outer boundary. We first expected that \( \partial K / \partial r < 0; \) that is, the value of \( K \) is greater near the inner boundary than that near the outer boundary because the buoyant production term \( P_{KB} \) decreases as \( r \) increases. Figure 4(a) shows that this is not the case. Nevertheless, the gradient \( -\Omega_0 \cdot \nabla K \) is positive (negative) in the northern (southern) hemisphere; this sign is the same as that expected under the assumption of \( \partial K / \partial r < 0. \) Since this gradient is involved in the \( H \) equation as the diffusion term, it leads to the turbulent helicity in the same sign as shown in Fig. 4(b). The peak of \( H \) is located where the gradient of \( K \) is steep. We can see that the absolute value of \( H \) is very large. This suggests that the length scale of \( \omega^2 \) is very short compared to \( R_{\text{out}} \) so that \( H \) is
much greater than \( K/R_{\text{out}} \). As the feature of geomagnetic field the ratio of the magnetic energy to the kinetic energy is important.\(^{29,47} \)

The value of \( K \) in Fig. 4(a) is about 800; it is ten times larger than the mean magnetic field energy whose volume average is about 80 as shown in Fig. 2. In the present model the ratio of \( \langle b^2 \rangle /2 \) to \( \langle u^2 \rangle /2 \) is unknown. The ratio of \( B^2/2 \) to \( U^2/2 \) needs to be examined in future work without the assumption of zero mean velocity.

In order to understand the reason for the profiles of \( K \) and \( H \) we examine their transport equations. Figure 5(a) shows the terms in the \( K \) equation as functions of \( r \) at \( \theta = \pi/2 \) or on the equator. As mentioned in the previous section the value of the buoyant production term is given in advance. The dissipation term is nearly balanced by the buoyant production term. Although the magnetic production term \( P_{KM} \) has a slightly negative value at \( r=0.4 \), it increases to a large positive value at \( r=0.85 \); it is greater than that of the buoyant production term. At this location the current density \( J_r \) or the gradient of \( B_\phi \) in the \( \theta \) direction is very large as shown in Fig. 3(a). This leads to a large value of \( \beta J^2 \) involved in \( P_{KM} \); a large amount of energy is cascaded from \( B^2/2 \) to \( K \). Therefore, the peak of \( K \) located near the outer boundary in Fig. 4(a) is caused by the magnetic production term. The turbulent diffusion term is nearly balanced by the magnetic production term. This means that the turbulent energy is transferred from the peak to other locations. Figure 5(b) shows the terms in the \( H \) equation at \( \theta = \pi/4 \) or at the middle of the northern hemisphere. The magnetic production term is negligibly small in the \( H \) equation. The diffusion term due to the mean vorticity or the system rotation, \( -\Omega_r \cdot \nabla K \), produces a positive value of \( H \). The dissipation term is not very large. Instead, the turbulent diffusion term is negative and is balanced by the diffusion term due to the system rotation. This result clearly shows that the system rotation and the gradient of \( K \) in the axial direction are important factors for the turbulent helicity production.

The resulting turbulent helicity causes the \( \alpha \) dynamo effect. We examine this effect and the turbulent diffusivity in the balance of the electric field. Substituting Eq. (4) into Eq. (3) with \( U=0 \) we have

\[
E = - \alpha B + \beta J + \eta J.
\]  

Figure 6 shows terms of the toroidal and radial components of the electric field at \( \theta = \pi/4 \). The last term on the right-hand side of Eq. (31) is not plotted because \( \eta \) is very small. In a steady state the toroidal field \( E_\phi \) should vanish; otherwise the poloidal magnetic field varies in time. Figure 6(a) clearly shows that the \( \alpha \) and \( \beta \) terms balance each other. Since the turbulent diffusivity causes the mean magnetic field to be diffused and weak, we can see that the \( \alpha \) term actually contributes to a dynamo mechanism. The poloidal field \( (E_r, E_\theta) \) can have a nonzero value because irrotational field satisfying \( (\nabla \times E)_\phi = 0 \) is allowed as a steady state. In Fig. 6(b) \( E_\phi \) shows positive values. Nevertheless, the \( \alpha \) term shows opposite sign compared to the \( \beta \) term; the former acts as a dynamo.

Since the value of \( \beta \) is of order unity, the molecular diffusivity \( \eta = 1.15 \times 10^{-3} \) is negligibly small. However, it does not mean that the turbulent diffusivity is always much greater than \( \eta \). The magnitude of \( \beta \) depends on the definition of averaging adopted. For example, in the LES the GS field is explicitly solved whereas the effect of the SGS field on the GS field is modeled; the turbulent diffusivity can be expressed as \( \beta \propto u_{SGS} \Delta \) where \( u_{SGS} \) is the intensity of the SGS velocity and \( \Delta \) is the filter width. Therefore, if we use a finer grid then \( \beta \) decreases; the value of \( \eta \) is not always negligible compared to \( \beta \) in three-dimensional LES. The \( \eta \) term in Eq. (31) can be dominant if the grid is so fine that the local turbulence can be appropriately resolved as shown by Bra-
ginsky and Meytis. On the other hand, the Reynolds-averaged model roughly corresponds to an LES with very large $\Delta$. To be more specific, axisymmetric modes are treated as the mean field whereas all nonaxisymmetric modes are considered the fluctuation. The effect of the fluctuation on the mean field in the Reynolds-averaged model is greater than three-dimensional LES; this is why the value of $\beta$ is very large in the present simulation.

So far we showed profiles for $V_0=3\times10^5$ in case 1. Here, we examine the case of slower rotation. Figure 7 shows profiles of the magnetic field at $t=1.5$ for $V_0=3\times10^4$ in case 1. Compared to the case of $V_0=5\times10^5$ the intensity of the magnetic field is low. The peak location also changes; it shifts to a point at high latitude and close to the rotation axis. This change suggests that the peak location of the $\alpha$ dynamo effect also shifts similarly. Figure 8 shows the contour plots of the turbulent energy $K$ and the turbulent helicity $H$ for $V_0=3\times10^4$. In Fig. 8(a) the peak of $K$ is located at a point on the equator near the inner boundary unlike that in Fig. 4(a). This difference is because the magnetic production term in the $K$ equation for $V_0=3\times10^4$ is not as large as that for $V_0=5\times10^5$ and the buoyant production term proportional to $r^{-1}$ contributes more. The region of large value of $-\Omega_0\cdot\nabla K$ shifts closer to the rotation axis compared to Fig. 4(a). As a result, the peak location of $H$ in Fig. 8(b) moves to high latitude and the $\alpha$ dynamo effect is maximum there.

As shown in Fig. 2 the magnetic energy for $\Omega_0=5\times10^5$ in case 2 is greater than that in case 1. Figure 9 shows profiles of the magnetic field at $t=1.8$ for $\Omega_0=5\times10^5$ in case 2. The peak value is greater than that in case 1 for both toroidal and poloidal fields. The peak location shifts slightly toward the outer boundary. The toroidal field is different from that in case 1 in that regions of opposite sign appear near the rotation axis. These profiles suggest that the $\alpha$ dynamo caused by the system rotation is greater in case 2 than in case 1.

Values of seven model constants are different between cases 1 and 2; it is not clear which constant is mainly responsible for the difference in profile. Here, we examine the magnetic field by changing only one or two of the model constants compared to case 1. Figure 10 shows the time evolution of the volume averaged energy of the mean magnetic field for $\Omega_0=5\times10^5$ for four runs in addition to case 1; in each run one or two of the model constants have different...
Moreover, in the case of \( C_{H1} = 1.8 \) the magnetic energy is nearly independent of the magnetic energy is nearly the same as that in case 1. This change is because the increase in the production term in the \( \varepsilon \) equation causes the increase in \( \varepsilon \) itself and the decrease in the \( D \) term. Since the \( \alpha \) term is nearly independent of \( \varepsilon \) as shown in Eq. (32), the \( \alpha \) dynamo effect is relatively enhanced to produce more magnetic energy. Therefore, the constants \( C_{H1} = 0.14\), \( C_{H2} = 0.14\), and \( C_{H1} = 1.8 \) are responsible for the difference between cases 1 and 2. The effect of \( C_{H1} \) decreasing the \( D \) term is slightly greater than that of \( C_{H2} \) increasing the \( D \) term; as a result the magnetic field is more diffused and the energy is smaller than that in case 1. On the other hand, in the case of \( C_{H1} = 1.8 \), the magnetic energy increases to more than twice the value in case 1. The difference between cases 1 and 2 is mainly caused by the coefficients in the turbulent diffusivity term in the turbulent electromotive force. Considering the realizability condition for the turbulent electromotive force, we introduced the correction coefficients in the dynamo and turbulent diffusivity terms; this correction enables us to obtain a steady-state solution even in the case of rapidly rotating systems such as the Earth.

We examined profiles of the magnetic field, the turbulent energy, and the turbulent helicity as well as their transport equations. The magnetic field profiles show a simple structure similar to results of a three-dimensional simulation with the stress-free boundary conditions. The dipole moment is close to the observed value of the Earth. The peak of the turbulent energy is located on the equator near the outer boundary. It was shown that the turbulent energy gradient in the direction of the rotation axis produces the turbulent helicity and leads to the \( \alpha \) dynamo effect. In the case of slower rotation, the magnetic energy decreases and the peak of the magnetic field shifts toward the rotation axis. We also examined the dependence on model constants. It was shown that the difference between cases 1 and 2 is mainly caused by the coefficients in the turbulent diffusivity term in the turbulent electromotive force and in the production term in the \( \varepsilon \) equation.

Since many model constants are involved in the present model, more work needs to be done to optimize their values and to improve model expressions. Moreover, we should solve the mean velocity and temperature equations to construct a self-consistent turbulence model. Nevertheless, we believe that the present work is important as a first step toward developing a general turbulence model that can predict the magnetic field in astrophysical/geophysical and engineering MHD flows.

**V. CONCLUSIONS**

To assess and improve the Reynolds-averaged turbulence model for MHD flows, we carried out numerical simulation of the magnetic field in a rotating spherical shell. In the three-equation model, in addition to the mean magnetic field, we treat the transport equations for the turbulent energy, its dissipation rate, and the turbulent helicity. The turbulent electromotive force is expressed as the sum of the \( \alpha \) dynamo term and the turbulent diffusivity term; they are modeled in terms of the three model variables. Considering the realizability condition for the turbulent electromotive force, we introduced the correction coefficients in the dynamo and turbulent diffusivity terms; this correction enables us to obtain a steady-state solution even in the case of rapidly rotating systems such as the Earth.

We examined profiles of the magnetic field, the turbulent energy, and the turbulent helicity as well as their transport equations. The magnetic field profiles show a simple structure similar to results of a three-dimensional simulation with the stress-free boundary conditions. The dipole moment is close to the observed value of the Earth. The peak of the turbulent energy is located on the equator near the outer boundary. It was shown that the turbulent energy gradient in the direction of the rotation axis produces the turbulent helicity and leads to the \( \alpha \) dynamo effect. In the case of slower rotation, the magnetic energy decreases and the peak of the magnetic field shifts toward the rotation axis. We also examined the dependence on model constants. It was shown that the difference between cases 1 and 2 is mainly caused by the coefficients in the turbulent diffusivity term in the turbulent electromotive force and in the production term in the \( \varepsilon \) equation.

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